

Global heat-seeded Picard convergence near a nonlinear periodic Navier–Stokes flow

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Abstract

We consider the unforced three-dimensional incompressible Navier–Stokes equations on the normalized 2π -periodic torus at viscosity one. For $W = (\sin z, \sin x, \sin y)$ and every smooth solenoidal, real, mean-zero perturbation satisfying $\sum_{k \neq 0} |k|^{-1} |\hat{h}(k)| \leq 10^{-4}$, we certify convergence of the ordinary Picard sequence starting from the complete heat flow $e^{t\Delta}(W+h)$. Convergence holds on the whole half-line in $\|f\|_{\mathcal{X}} = (\int_0^\infty (\sum_{k \neq 0} |\hat{f}(k)|)^2 dt)^{1/2}$, with error at iteration $n \geq 3$ at most $6(9/10)^{n-3}$ and $\int_0^\infty A_0(u(t))^2 dt < 9$, where $A_0(f) = \sum_{k \neq 0} |\hat{f}(k)|$. The proof retains the full causal history in a finite block norm and verifies entry of the actual heat-seeded third iterate into one invariant set. Separate critical-Fourier comparison estimates establish larger explicit global-regularity neighborhoods. A validated finite-reference certificate also gives the fixed- W all-cutoff bound $\sup_N \int_0^\infty \|\nabla u_N\|_2^4 dt < 21/2$. The stated quantitative estimates are certified and are not claimed optimal. Exact finite certificates and their analytic connection to the full PDE are specified in the accompanying reproducibility material.

Keywords: Navier–Stokes equations; critical Fourier norms; Picard iteration; a posteriori estimates; computer-assisted proof.

1 Introduction

A global regularity certificate for a specified Navier–Stokes datum and a convergence theorem for a prescribed iteration answer different questions. The first controls the exact solution. The second must also control the iterates generated by a fixed starting rule. This paper establishes both kinds of result for explicit neighbourhoods of a nonlinear three-dimensional periodic field, while keeping their respective hypotheses separate.

We work with the unforced incompressible Navier–Stokes equations on $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$, with normalized spatial measure, viscosity one, and zero spatial mean. The centre is

$$W(x, y, z) = (\sin z, \sin x, \sin y).$$

Its active wave vectors span three dimensions, and its projected quadratic nonlinearity is nonzero. In particular, its evolution is not simply its heat evolution. For a mean-zero field we use the Fourier norms

$$A_s(f) = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} |k|^s |\hat{f}(k)|_2, \quad \|f\|_{\mathcal{X}} = \left(\int_0^\infty A_0(f(t))^2 dt \right)^{1/2}.$$

Here $|\cdot|_2$ is the Hermitian Euclidean norm, also written $|\cdot|$ below. We call $\int_0^\infty A_0(u(t))^2 dt$ the solution action. Unless stated otherwise, perturbations are smooth, real, mean-zero and divergence-free. They may have infinitely many Fourier modes. Smallness in A_{-1} does not impose a uniform bound on their higher Sobolev norms.

Our main result concerns the ordinary heat-seeded Picard sequence for initial data $W + h$. With $S(t) = e^{t\Delta}$ and the usual Leray-projected Duhamel operator Q , this sequence is

$$v_h^{(0)} = S(W + h), \quad v_h^{(n+1)} = S(W + h) - Q(v_h^{(n)}, v_h^{(n)}).$$

For $A_{-1}(h) \leq 10^{-4}$, it converges in $\mathcal{X}$ to the global smooth solution u_h , and

$$\|v_h^{(n)} - u_h\|_{\mathcal{X}} \leq 6(9/10)^{n-3}, \quad n \geq 3.$$

The proof certifies entry of the actual third iterate into a common invariant set. A finite nonnegative lower-triangular matrix then controls subsequent errors on a partition of the entire time half-line. The same construction gives quantitative dependence on the initial datum and a uniform Galerkin continuation bound. Thus the asserted rate concerns the specified ordinary iteration, rather than an iteration initialized at an independently chosen approximation.

A second result gives larger regularity neighbourhoods for data $aW + h$. Among the certified regions are

$$\begin{aligned} |a| &\leq 1, & A_{-1}(h) &\leq 1/10, \\ |a| &\leq 11/10, & A_{-1}(h) &\leq 1/40, \\ |a| &\leq 8/7, & A_{-1}(h) &\leq 1/100. \end{aligned}$$

These are separate regions, not the rectangle obtained by combining their largest amplitude and largest radius. They yield global smoothness and explicit continuation bounds, but the Picard convergence assertion above is not extended to their union.

Finally, a validated spherical Galerkin certificate at cutoff three provides a complementary estimate for the unperturbed datum. If u_M denotes an exact spherical Fourier–Galerkin solution starting from the projection of W , the certificate gives

$$\sup_M \sup_{t \geq 0} \|\nabla u_M(t)\|_2^2 < 14, \quad \sup_M \int_0^\infty \|\nabla u_M(t)\|_2^4 dt < 21/2.$$

The finite-dimensional residual includes every exterior mode generated by the reference trajectory. An analytic tail completes the estimate after the validated finite interval. These particular numerical bounds concern W ; they are not assigned unchanged to perturbed data.

The proof consists of analytic estimates together with finite arithmetic certificates. The certificates verify specific inequalities needed by the analytic argument; they do not replace the argument connecting Galerkin estimates to the full PDE. Our scope is the stated families and conclusions. We make no claim concerning unrestricted smooth initial data, optimal perturbation radii, or the earliest occurrence of the general methods used here.

For perspective, the zero-reference specialization of Lemma 3.2 gives global smoothness for smooth, real, mean-zero, solenoidal data with $A_{-1}(u_0) < 2$; its Galerkin-to-PDE justification is given after that lemma. Separately, Lemma 2.2 gives the elementary Picard contraction condition $A_{-1}(u_0) < 1$, since $4c\|Su_0\|_{\mathcal{X}} \leq A_{-1}(u_0)$, with $c = 1/(2\sqrt{2})$. The closed ball centred at zero with radius $2\|Su_0\|_{\mathcal{X}}$ is invariant and strictly contractive and contains the heat seed. These are sufficient conditions, not optimal thresholds. Since $A_{-1}(W) = 3$, the zero-reference criterion covers $aW + h$ whenever $3|a| + A_{-1}(h) < 2$, and an entire closed amplitude–radius class whenever $3a_* + r < 2$. By contrast, every datum of the main theorem satisfies $A_{-1}(W + h) \geq 3 - 10^{-4} > 2$.

The reported action, stability and Galerkin bounds are certified estimates; the perturbation radii are sufficient admissible radii. Neither the bounds nor the radii are claimed optimal or representative of typical numerical behaviour.

1.1 Relation to existing results

Lei and Lin [3] established global small-data mild solutions in a critical Fourier space on $\mathbb{R}^3$. Ambrose, Lopes Filho and Nussenzweig Lopes [4] prove periodic small-data well-posedness using a whole-half-line contraction in Fourier–Wiener spaces. Their argument also gives convergence from the ordinary heat seed in their space, hence in $\mathcal{X}$: this last implication follows from

$$\|f\|_{\mathcal{X}}^2 \leq \left(\sup_{t \geq 0} A_{-1}(f(t)) \right) \int_0^\infty A_1(f(t)) dt.$$

Thus ordinary heat-seeded iteration and whole-half-line convergence are established principles. Here we supply the explicit neighbourhood of W , certified entry of the actual third iterate, and the stated uniform bounds and rate.

Critical Fourier perturbation theory is also established. Benameur’s Theorem 1.4 in [5] gives quantitative stability in $\mathcal{X}^{-1}(\mathbb{R}^3)$ around an already global solution, with a perturbation threshold depending exponentially on a time-integrated Fourier norm of that solution. Its stated domain is $\mathbb{R}^3$. Our periodic certificate supplies a specified reference, explicit verified bounds, and a convergence conclusion for a prescribed iteration. We do not present stability of global solutions as a new principle. Explicit high-Sobolev stability estimates for smooth periodic solutions are also part of the literature; see [9].

The use of approximate trajectories and invariant error tubes has precedent in the literature. Morosi and Pizzocchero [6], Proposition 3.7 and Lemmas 4.8–4.9, construct solutions by Peano–Picard iteration from a suitable approximate seed. Their existence conclusion holds on the whole interval on which the control inequality is satisfied and can therefore be global; their stated iteration estimates give uniform convergence on each compact subinterval. Their later Sobolev control theorem [7], Proposition 4.5, bounds an exact periodic Euler or Navier–Stokes solution through a scalar control inequality. These results provide the comparison framework. The task here is the explicit verification of the stated data classes and of the heat-seed entry and whole-half-line convergence estimates, not the introduction of a posteriori analysis itself.

Morosi, Pernici and Pizzocchero [8] use high-order Reynolds expansions and a posteriori Sobolev control for structured periodic initial data. Their Proposition 2.3 gives the analytic implication from a control equation to exact-solution bounds. The paper explicitly distinguishes its numerical control-equation calculations from certified computer-assisted proofs in footnote 4. A Reynolds expansion is not the ordinary heat-seeded Picard sequence. Accordingly, the relevant comparisons concern the approximation rule, function space, data class and certification standard, rather than only a displayed amplitude or Reynolds number.

Silva [10], Theorem 2.2, states a global smoothness theorem on the same normalized torus at viscosity one for the cyclic-shear class

$$\mathcal{C}_S = \{aU + h : 0 \leq a \leq 7/10, A_2(h) \leq 1/100\}, \quad U = (\sin y, \sin z, \sin x),$$

where the perturbations satisfy the same smoothness, reality, mean and divergence conditions. The theorem also supplies a residual-action estimate uniform over cubical Galerkin cutoffs. That action estimate is a distinct conclusion from the bounds stated here.

For clarity, the inclusion between the corresponding regularity classes can be checked directly. Let P be the orthogonal matrix interchanging the first two coordinates and define

$$(\mathcal{T}f)(x) = Pf(P^T x).$$

This simultaneous coordinate and velocity-component permutation preserves the Navier–Stokes equations and every A_s norm, and $\mathcal{T}U = W$. Since $|k| \geq 1$ for every nonzero integer wave vector,

$$A_{-1}(\mathcal{T}h) \leq A_2(\mathcal{T}h) = A_2(h).$$

Consequently, $\mathcal{TC}_S$ is contained in our region

$$\mathcal{C}_E = \{aW + h : |a| \leq 8/7, A_{-1}(h) \leq 1/100\}.$$

The inclusion is strict. Indeed, $W \in \mathcal{C}_E$, whereas any representation $W = bW + h$, with $0 \leq b \leq 7/10$, necessarily has

$$A_2(h) = A_2((1-b)W) = 3(1-b) \geq 9/10 > 1/100.$$

Thus $W \notin \mathcal{TC}_S$, even allowing an alternative amplitude decomposition.

This comparison establishes a strict enlargement of this specified cyclic-shear class for the conclusion of global PDE regularity. It neither compares the numerical residual-action constants nor transfers estimates between cubical and spherical cutoffs. It also does not assert inclusion of Silva’s separate ABC, TG3 or coefficient families. The main Picard theorem supplies a different additional conclusion on its own smaller neighbourhood. Attribution and development history are separate from these mathematical inclusion statements.

2 The ordinary heat-seeded iteration near W

We give the analytic argument and specify the finite certificate on which its numerical constants depend. The certificate evaluates known Fourier and exponential-polynomial expressions; it does not approximate an unknown Navier–Stokes trajectory. Its exact rational data and checkers form part of the accompanying reproducibility material.

2.1 Conventions and statement

Let $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$, with measure $(2\pi)^{-3} dx$. Fourier coefficients are denoted by $\widehat{f}_k$; $|\widehat{f}_k|$ is the Hermitian Euclidean norm on $\mathbb{C}^3$. All vector fields are real, mean-zero and solenoidal. Thus $\widehat{f}_{-k} = \overline{\widehat{f}_k}$ and $k \cdot \widehat{f}_k = 0$, where the latter product is bilinear. For $k \neq 0$, put $P_k = I - k \otimes k / |k|^2$, and write $\mathbb{P}$ for the Leray projection. Define

$$A_s(f) = \sum_{k \neq 0} |k|^s |\widehat{f}_k|, \quad S(t) = e^{t\Delta}, \quad A = -\Delta, \quad (1)$$

$$B(f, g) = \mathbb{P}((f \cdot \nabla)g), \quad B_s(f, g) = \tfrac{1}{2}\{B(f, g) + B(g, f)\}, \quad (2)$$

$$Q(f, g)(t) = \int_0^t S(t-s)B_s(f, g)(s) ds, \quad \mathcal{H}r(t) = \int_0^t S(t-s)r(s) ds. \quad (3)$$

The viscosity is one and there is no external force. The projected equation is $u' + Au + B(u, u) = 0$, and its mild form is $u = Su_0 - Q(u, u)$. In particular the sign convention in (2) is positive convection. Let

$$\mathcal{X}(I) = L^2(I; A_0), \quad \|f\|_{\mathcal{X}(I)}^2 = \int_I A_0(f(t))^2 dt, \quad \mathcal{X} = \mathcal{X}((0, \infty)). \quad (4)$$

Precisely, this is the closed real, mean-zero, solenoidal subspace of $L^2(I; \ell^1(\mathbb{Z}^3 \setminus \{0\}; \mathbb{C}^3))$.

For $N > 0$, P_N retains the modes $0 < |k| \leq N$. The Galerkin solution associated with u_0 solves

$$u'_N + Au_N + P_NB(u_N, u_N) = 0, \quad u_N(0) = P_Nu_0. \quad (5)$$

Write

$$E_0 = \|u_0\|_2^2, \quad X_0 = \|\nabla u_0\|_2^2, \quad B_*(u_0) = \sup_{N>0} \int_0^\infty \|\nabla u_N(t)\|_2^4 dt. \quad (6)$$

Theorem 2.1 (Ordinary heat-seeded convergence). *Let $W = (\sin z, \sin x, \sin y)$ and $u_0 = W + h$, where h is smooth, real, mean-zero and divergence-free, and*

$$A_{-1}(h) \leq \varepsilon := 10^{-4}. \quad (7)$$

There is a unique global smooth solution u_h with datum u_0 . The ordinary Picard sequence

$$v_0^h = S(W + h), \quad v_{n+1}^h = T_h(v_n^h), \quad T_h(f) = S(W + h) - Q(f, f) \quad (8)$$

converges to u_h in $\mathcal{X}$ on the entire half-line, with

$$\|v_n^h - u_h\|_{\mathcal{X}} \leq 6(9/10)^{n-3} \quad (n \geq 3), \quad \int_0^\infty A_0(u_h)^2 dt < 9. \quad (9)$$

For two perturbations satisfying (7),

$$\|u_h - u_g\|_{\mathcal{X}} \leq 4757 A_{-1}(h - g). \quad (10)$$

Moreover,

$$B_*(u_0) \leq \frac{E_0 X_0}{2} e^9. \quad (11)$$

There is no restriction on the Fourier support of h .

The convergence rate concerns $\mathcal{X}$, not an H^1 rate. The quantities E_0 and X_0 in (11) belong to the full datum; the critical ball (7) does not bound them uniformly. The theorem is distinct from the larger regularity regions proved elsewhere in this paper: their existence conclusions do not imply convergence of (8) throughout those regions.

2.2 The heat bilinear estimate and the initial trace

Lemma 2.2 (Complex pair bound and heat estimate). *The space $\mathcal{X}$ is complete, and*

$$\|Q(f, g)\|_{\mathcal{X}} \leq c \|f\|_{\mathcal{X}} \|g\|_{\mathcal{X}}, \quad c = \frac{1}{2\sqrt{2}}. \quad (12)$$

The same bound holds for output projections and for zero-extended temporal block inputs. For $A_{-1}(a) < \infty$,

$$\|Sa\|_{\mathcal{X}} \leq \frac{A_{-1}(a)}{\sqrt{2}}. \quad (13)$$

The operator Q maps $\mathcal{X} \times \mathcal{X}$ continuously into $C([0, \infty); A_{-1})$, with zero initial trace, and

$$\sup_{t \geq 0} A_{-1}(Q(f, g)(t)) \leq \frac{1}{2} \|f\|_{\mathcal{X}} \|g\|_{\mathcal{X}}. \quad (14)$$

Proof. For a real unit vector n , write $a = \alpha n + a_\perp$ and $b = \beta n + b_\perp$, allowing complex components. Then

$$\begin{aligned} |P_{n^\perp}[(n \cdot a)b + (n \cdot b)a]| &= |\alpha b_\perp + \beta a_\perp| \\ &\leq |\alpha| |b_\perp| + |\beta| |a_\perp| \leq |a| |b|. \end{aligned}$$

The last inequality is Cauchy–Schwarz for the two nonnegative pairs. Using input solenoidality to place derivatives on the output, and reindexing the second symmetric convolution, gives

$$|\widehat{B_s(f, g)}_k| \leq \frac{|k|}{2} \sum_{p+q=k} |\widehat{f}_p| |\widehat{g}_q|, \quad k \neq 0. \quad (15)$$

The zero output coefficient vanishes. For each nonzero mode,

$$\|\mathbf{1}_{t \geq 0} |k| e^{-|k|^2 t}\|_{L_t^2} = 1/\sqrt{2}.$$

Minkowski in the Fourier sum, Young’s L_t^1 – L_t^2 convolution inequality for this individual kernel, and time Cauchy–Schwarz prove (12). This argument keeps the modewise kernel until its time norm is taken; it does not integrate a pointwise supremum over modes. For the homogeneous term, Minkowski gives

$$\|Sa\|_{\mathcal{X}} \leq \sum_{k \neq 0} |\widehat{a}_k| \left(\int_0^\infty e^{-2|k|^2 t} dt \right)^{1/2} = \frac{1}{\sqrt{2}} A_{-1}(a).$$

Dividing (15) by $|k|$ and using contraction of S in A_{-1} proves

$$A_{-1}(Q(f, g)(t)) \leq \frac{1}{2} \int_0^t A_0(f(s)) A_0(g(s)) ds,$$

and hence (14). The forcing belongs to $L_t^1 A_{-1}$. Strong continuity of the heat semigroup on the weighted ℓ^1 space, together with approximation of that forcing by simple functions, proves continuity and the zero trace. The same reasoning gives the trace a for Sa . One may first make all computations on finite Fourier sums and compact time supports, then pass by density. Completeness follows from the Banach-space definition of $\mathcal{X}$. $\square$

We will also use uniqueness among mild solutions in $\mathcal{X}$ with the same datum. Indeed, subdivide time into finitely many intervals, including a final infinite interval, so that $c(\|u\|_{\mathcal{X}(I)} + \|v\|_{\mathcal{X}(I)}) < 1$ on each one. Their finite total squared norms permit such a subdivision. If the two solutions agree on the preceding intervals, the entire preceding Duhamel history cancels in their difference. Equation (12) on the next interval then forces equality there. This proves uniqueness on the whole half-line without resetting either solution.

2.3 Explicit references and the complete heat defect

For a smooth reference w with initial trace W , define $r_w = w' + Aw + B_s(w, w)$. Direct integration gives

$$w - \mathcal{H}r_w = SW - Q(w, w). \quad (16)$$

Thus, with $z = \mathcal{H}r_w$ and $\tilde{w} = w - z$,

$$r_{\tilde{w}} = -2B_s(w, z) + B_s(z, z), \quad z(0) = 0. \quad (17)$$

These are identities for the full residual, including its low modes.

Let $w_0 = SW = e^{-t}W$, $w_{j+1} = SW - Q(w_j, w_j)$, and put

$$\begin{aligned} F &= B(W, W) = (\sin y \cos z, \sin z \cos x, \sin x \cos y), \\ C &= \frac{1}{2}(\sin z \cos 2y, \sin x \cos 2z, \sin y \cos 2x), \\ G &= B(W, F) + B(F, W) = -W/2 + C, \quad H = B(F, F). \end{aligned} \quad (18)$$

The Leray projection is part of these definitions. In particular $F \neq 0$. Define the nonnegative causal heat profiles

$$\begin{aligned} L_1 &= \frac{1}{4}\{e^{-t} - (1 + 2t)e^{-3t}\}, \\ L_5 &= \frac{1}{4}\{e^{-5t} + (2t - 1)e^{-3t}\}, \\ L_6 &= \frac{1}{4}\{(2t^2 - 2t + 1)e^{-4t} - e^{-6t}\}. \end{aligned} \quad (19)$$

They vanish at zero and satisfy $(\partial_t + 1)L_1 = (\partial_t + 5)L_5 = te^{-3t}$ and $(\partial_t + 6)L_6 = t^2e^{-4t}$. Consequently

$$\begin{aligned} w_1 &= e^{-t}W - te^{-2t}F, \\ w_2 &= (e^{-t} - L_1/2)W - te^{-2t}F + L_5C - L_6H. \end{aligned} \quad (20)$$

The support of w_2 lies in $|k|^2 \leq 6$. Its exact amplitude is

$$U_2 = A_0(w_2) = \frac{21}{8}e^{-t} + 3te^{-2t} + \frac{3}{2}te^{-3t} + \frac{3}{8}e^{-5t} + \sqrt{3}L_6. \quad (21)$$

This follows from its disjoint Fourier shells and the positive fundamental coefficient; it is not the sum of magnitudes of an arbitrarily signed representation.

For explicit residual linkage, set $x = e^{-t}$, $y = te^{-2t}$, $p = L_1/2$, $q = L_5$, $r = L_6$, and $K_{AB} = B(A, B) + B(B, A)$. The scalar profiles p, q, r are local to this subsection; the later block argument uses p for a positive vector and q for a contraction factor. Then

$$\begin{aligned} r_2 &= (p^2 - 2xp)F + ypG + (x - p)qK_{WC} - (x - p)rK_{WH} \\ &\quad - yqK_{FC} + yrK_{FH} + q^2B(C, C) - qrK_{CH} + r^2B(H, H). \end{aligned} \quad (22)$$

Define

$$z_3 = \mathcal{H}r_2, \quad w_3 = w_2 - z_3, \quad z_4 = \mathcal{H}\{-2B_s(w_2, z_3) + B_s(z_3, z_3)\}. \quad (23)$$

Thus w_3 is the third unperturbed heat-seeded iterate and $T_h(w_3) - w_3 = Sh - z_4$. The finite certificate constructs all signed coefficients of z_3 (342 nonzero modes); the conservative support bound $|k|^2 \leq 24$ contains w_3 . It constructs the signed mixed contribution $-2\mathcal{H}B_s(w_2, z_3)$ to z_4 exactly. The quadratic contribution $\mathcal{H}B_s(z_3, z_3)$ is bounded as a whole, not discarded or represented as part of the mixed-mode count.

The reference calculations admit a real rational storage convention without restricting the perturbation class. Under simultaneous inversion of the spatial coordinates, $W(-\mathbf{x}) = -W(\mathbf{x})$. If f and g are odd, then ∇g is even, so $(f \cdot \nabla)g$ is odd; the Leray projection and heat semigroup preserve this parity. Thus all the W -based reference and defect fields constructed here are real and odd. Reality and oddness together imply

$$\widehat{f}_k = i\xi_k, \quad \xi_k \in \mathbb{R}^3, \quad \xi_{-k} = -\xi_k, \quad |\widehat{f}_k| = |\xi_k|.$$

For two such fields, the positive-convection convention (2) reads

$$\widehat{B(f, g)}_k = -iP_k \sum_{p+q=k} (\xi_p \cdot q)\eta_q, \quad \widehat{g}_q = i\eta_q.$$

This explains the minus sign in the stored real-vector convolution. The programs store ξ_k , or the rational coefficients of its real exponential-polynomial profile; their real Euclidean norm is exactly the Hermitian norm of the actual Fourier coefficient. This convention is used only for the explicit reference and defect calculations. The perturbation h , its Picard iterates, and the fixed-point space are not assumed odd; their estimates use the general complex-vector inequality (15).

Here is the analytic basis of the finite enclosures. Heat convolution of $t^m e^{-\alpha t}$ at frequency $|k|^2 = \lambda$ is $e^{-\lambda t} \int_0^t s^m e^{(\lambda-\alpha)s} ds$, a rational exponential polynomial for the integer rates occurring here. Products of mode profiles therefore have exact factorial moments. Their tails use

$$\int_a^\infty t^m e^{-\lambda t} dt = e^{-\lambda a} \sum_{j=0}^m \frac{m! a^j}{j! \lambda^{m-j+1}}. \quad (24)$$

After combining signed terms within each mode, Minkowski bounds its $\mathcal{X}(I)$ norm by the sum of square roots of these squared moments. Endpoint exponentials and radicals are enclosed outward by rational series and integer-square-root bounds.

For the quadratic part of z_4 , we construct a pointwise upper profile explicitly. Let $\Phi_1, \dots, \Phi_9$ be the nine spatial fields in the order displayed in (22). Then $r_2 = \sum_{j=1}^9 \sigma_j g_j \Phi_j$, where

$$\begin{aligned} (\sigma_1, \dots, \sigma_9) &= (-1, 1, 1, -1, -1, 1, 1, -1, 1), \\ (g_1, \dots, g_9) &= (p(2x-p), yp, (x-p)q, (x-p)r, yq, yr, q^2, qr, r^2). \end{aligned} \quad (25)$$

The heat-convolution definitions of L_1, L_5, L_6 give $x, y, p, q, r \geq 0$, and the explicit formula for L_1 gives $0 \leq p = L_1/2 \leq x/8$. Thus every g_j is nonnegative on $[0, \infty)$. For every shell present in Φ_j , choose a rational number

$$\eta_{j\lambda}^+ \geq \sum_{|k|^2=\lambda} |\widehat{\Phi}_{jk}|, \quad \lambda \geq 1,$$

by summing outward upper bounds for the individual square roots. Define

$$\psi_{j\lambda}(t) = \int_0^t e^{-\lambda(t-s)} g_j(s) ds, \quad Z(t) = \sum_{j,\lambda} \eta_{j\lambda}^+ \psi_{j\lambda}(t). \quad (26)$$

All these sums are finite. Positivity of the scalar heat kernel implies $\psi_{j\lambda} \geq 0$. Since $\widehat{z}_{3,k} = \sum_j \sigma_j \widehat{\Phi}_{jk} \psi_{j,|k|^2}$, the triangle inequality, first over fields and then over shells, gives $A_0(z_3(t)) \leq Z(t)$. No disjointness of the fields' Fourier supports is assumed. This is the certificate's 31-term field-shell profile, stored under `pointwise_A0_upper_profile` in `third_heat_defect_static_result.json`. Its positivity follows from (26), even though the expanded exponential-polynomial coefficients can have either sign. For any $[a, b]$, including $b = \infty$, the same modewise kernel calculation as above gives

$$\|\mathcal{H}B_s(z_3, z_3)\|_{\mathcal{X}(a,b)} \leq c \left\{ \int_0^a e^{-(a-s)} Z(s)^2 ds + \int_a^b Z(s)^2 ds \right\}. \quad (27)$$

In particular its earlier forcing is retained. This estimate uses only $|k| \geq 1$; the corrected residual need not stay at high frequencies.

2.4 A finite-block criterion for the unchanged global map

Use the 17 intervals $I_i = [a_i, b_i]$ with endpoints

$$0, \frac{1}{16}, \frac{1}{8}, \frac{3}{16}, \frac{1}{4}, \frac{5}{16}, \frac{3}{8}, \frac{7}{16}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}, 1, \frac{5}{4}, \frac{3}{2}, 2, 3, \infty. \quad (28)$$

Overlaps of endpoints have measure zero. For $j < i$, let k_{ij} be a rational upper enclosure of

$$\sup_{\lambda \geq 1} e^{-\lambda(a_i - b_j)} \sqrt{1 - e^{-2\lambda(b_i - a_i)}}; \quad k_{ii} = 1. \quad (29)$$

For the infinite block the square-root factor is one. Define rational s_i satisfying

$$s_i \geq \frac{1}{\sqrt{2}} \sup_{\lambda \geq 1} e^{-\lambda a_i} \sqrt{1 - e^{-2\lambda(b_i - a_i)}}. \quad (30)$$

Put

$$\kappa_i := \sup_{\lambda \geq 1} e^{-\lambda a_i} \sqrt{1 - e^{-2\lambda(b_i - a_i)}}, \quad \bar{\kappa}_i \in \mathbb{Q}, \quad \kappa_i \leq \bar{\kappa}_i \leq 1.$$

For $b_i = \infty$, the square-root factor is understood to be one; then $\kappa_i = e^{-a_i}$. In the first block $a_1 = 0$ and $\kappa_1 = \bar{\kappa}_1 = 1$. The function `heat_factor` in `perturbation_block_gate.py` constructs these rational upper enclosures. The certificate takes $s_i = 2\bar{c}\bar{\kappa}_i$, which satisfies (30) because $2\bar{c} > 1/\sqrt{2}$, with

$$\bar{c} = \frac{707106781187}{2000000000000} > c. \quad (31)$$

The first supremum in (30) is one, even for a very short first block. Arbitrarily high-frequency perturbations prevent a uniform square-root-of-duration gain there.

Lemma 2.3 (Complete source-block memory). *For $f, g \in \mathcal{X}$,*

$$\|Q(f, g)\|_{\mathcal{X}(I_i)} \leq c \sum_{j \leq i} k_{ij} \|f\|_{\mathcal{X}(I_j)} \|g\|_{\mathcal{X}(I_j)}. \quad (32)$$

Also $\|Sh\|_{\mathcal{X}(I_i)} \leq s_i A_{-1}(h)$.

Proof. Split the forcing into its source intervals. For $s \in I_j$, $j < i$, and $\lambda = |k|^2$, the exact output-block norm of the differentiated heat kernel is

$$\left(\int_{a_i}^{b_i} |k|^2 e^{-2|k|^2(t-s)} dt \right)^{1/2} = \frac{1}{\sqrt{2}} e^{-\lambda(a_i-s)} \sqrt{1 - e^{-2\lambda(b_i-a_i)}} \leq k_{ij}/\sqrt{2}.$$

Minkowski, (15), and source-time Cauchy–Schwarz give each summand of (32). On the diagonal, use the whole future kernel norm $1/\sqrt{2}$. Blocks later than i contribute nothing. The homogeneous heat estimate follows by the same Fourier Minkowski calculation with no derivative. Every earlier source block remains in the sum. $\square$

The suprema in (29)–(30) have elementary enclosures. For gap $g > 0$ and length $\ell > 0$, the unconstrained squared maximum over $\lambda \geq 0$ is $\ell(g + \ell)^{-1}(g/(g + \ell))^{g/\ell}$, with maximizer $\log(1 + \ell/g)/(2\ell)$. If that maximizer is at most one, the constrained maximum is attained at one. The zero-gap and infinite-length cases are treated directly. Rational logarithm, exponential, and square-root enclosures suffice; no floating-point maximization is part of acceptance.

Choose certified rational bounds

$$V_i \geq \|w_3\|_{\mathcal{X}(I_i)}, \quad d_i \geq \|z_4\|_{\mathcal{X}(I_i)}. \quad (33)$$

Here V_i can be the sum of the enclosed w_2 and z_3 block norms. Suppose positive rational radii ρ_i satisfy

$$d_i + \varepsilon s_i + \bar{c} \sum_{j \leq i} k_{ij} (2V_j \rho_j + \rho_j^2) \leq \rho_i. \quad (34)$$

The set

$$\mathcal{B} = \{f \in \mathcal{X} : \|f - w_3\|_{\mathcal{X}(I_i)} \leq \rho_i \text{ for all } i\} \quad (35)$$

is then invariant under every T_h in (7). Indeed, $T_h(w_3) - w_3 = Sh - z_4$, and expansion around w_3 followed by (32) proves (34)’s assertion. On this set, block distances are bounded by the nonnegative lower-triangular matrix

$$M_{ij} = 2\bar{c} k_{ij} (V_j + \rho_j) \ (j \leq i), \quad M_{ij} = 0 \ (j > i). \quad (36)$$

If $\max_i M_{ii} < 1$, choose q between this maximum and one and set $p = (qI - M)^{-1}\mathbf{1}$. Forward substitution gives $p > 0$ and $Mp = qp - \mathbf{1} \leq qp$. The norm $\max_i \|f\|_{\mathcal{X}(I_i)}/p_i$ is equivalent to $\mathcal{X}$ because there are finitely many blocks. The map T_h is a strict contraction of the closed set (35) in that complete norm. This is an all-time fixed point of the unchanged map, not a chain of restarted local iterations. Lemma 2.2 gives its trace $W + h$.

2.5 The finite certificate and entry of the actual heat seed

The invariant-set construction alone would only prove convergence from points already in $\mathcal{B}$. We must check entry for the particular seed in (8). Let $V_{n,i}$ bound $\|w_n\|_{\mathcal{X}(I_i)}$ for $n = 0, 1, 2$. The first two exact amplitudes are $3e^{-t}$ and $3e^{-t} + 3te^{-2t}$. Their squared tail integrals are

$$\begin{aligned} F_0(t) &= \frac{9}{2}e^{-2t}, \\ F_1(t) &= \frac{9}{2}e^{-2t} + (6t + 2)e^{-3t} + \left(\frac{9}{4}t^2 + \frac{9}{8}t + \frac{9}{32}\right)e^{-4t}. \end{aligned} \quad (37)$$

The $n = 2$ bounds use (21). Thus these prefix inputs have explicit analytic constructions independent of the unknown solution.

Set $E_{0,i} = \varepsilon s_i$ and choose outward rational upper bounds recursively so that, for $n = 0, 1, 2$,

$$E_{n+1,i} \geq \varepsilon s_i + \bar{c} \sum_{j \leq i} k_{ij}(2V_{n,j}E_{n,j} + E_{n,j}^2). \quad (38)$$

Subtracting the exact perturbed and unperturbed Picard equations and applying Lemma 2.3 proves inductively

$$\|v_n^h - w_n\|_{\mathcal{X}(I_i)} \leq E_{n,i}, \quad 0 \leq n \leq 3. \quad (39)$$

No earlier source time is removed in this recurrence.

Proposition 2.4 (Finite exact certificate for the stated constants). *For (28) and (31), the supplied rational data, constructed from (18)–(27), satisfy (29)–(34) and*

$$\max_i M_{ii} < \frac{3}{5}, \quad E_{3,i} < \rho_i \quad (1 \leq i \leq 17), \quad C_0 := \sum_i (V_i + \rho_i)^2 < 9. \quad (40)$$

With $E = E_3 + \rho$, there is a rational vector $p > 0$ satisfying

$$p \geq E, \quad Mp \leq \frac{9}{10}p, \quad \sum_i p_i^2 < 36. \quad (41)$$

For $b = (I - M)^{-1}s$, the rational data also satisfy

$$b_i = \frac{s_i + \sum_{j < i} M_{ij}b_j}{1 - M_{ii}} > 0, \quad \sum_i b_i^2 < 4757^2. \quad (42)$$

Certificate construction and acceptance. The following files specify the finite arithmetic proof; stage-source paths are relative to `proof_sources/W_research/` in the companion `Code.zip`. The replay-wrapper path is relative to that archive's root.

1. `12_second_correction/second_correction_fourier_check.py` and its result construct the projected fields and full second residual.
2. `13_iteration_convergence/third_heat_defect_static.py` and its result construct every signed z_3 mode, check its forced heat identity and initial trace, and supply the positive profile Z .
3. `14_unit_W_picard/fourth_heat_defect_static.py` and its result construct the mixed z_4 modes and include the entire quadratic contribution through (27).

4. `14_unit_W_picard/refined_W_block_gate_check.py` and `refined_W_block_gate_result.json` evaluate the block moment enclosures and complete memory factors. The replay wrapper `checks/replay_W_chain.py` places regenerated dependencies in a common `working_copy/` directory, where the Stage15 programs consume them using their relative paths.
5. `15_perturbed_picard/perturbation_block_gate.py` and `perturbation_block_gate_result.json` provide the rational V, d, k, s, ρ, M and check every inequality (34).
6. `15_perturbed_picard/perturbed_picard_prefix_static.py` and its result provide all E_0, E_1, E_2, E_3 , verify (38) with outward rounding, and check all 17 strict entry comparisons in (40).
7. `15_perturbed_picard/perturbed_stability_rate_static.py` and its result check the matrix formula, (41), (42), and the squared solution budget.

Where not explicitly named, “its result” means the correspondingly named `_result.json` file. All integer/rational comparisons are performed on the stored fractions; displayed decimals do not decide acceptance. The exponential-polynomial integrals use (24). Endpoint exponentials, logarithms used in memory factors, and radicals have explicit outward series bounds. Source hashes bind the imported rational data; the separate replay record identifies which generators and checkers were rerun.

For orientation only, the resulting values are

$$\begin{aligned}\max_i M_{ii} &= 0.5781572417201134\dots, \\ \max_i E_{3,i}/\rho_i &= 0.9127337584808773\dots, \\ C_0 &= 8.340335743815434\dots\end{aligned}$$

The rate vector may be constructed by forward substitution:

$$p_i = \max \left\{ E_i, \frac{\sum_{j<i} M_{ij} p_j}{9/10 - M_{ii}} \right\}.$$

Its domination and matrix inequalities follow directly from this formula; the finite squared-norm comparison in (41) is included in the exact certificate. Thus the proposition rests on specified finite identities, enclosures and inequalities, rather than a saved status string. It remains a computer-assisted finite certificate, not a proof-assistant formalization; the full rational arrays are supplied to permit inspection and reproduction. $\square$

By (39) and Proposition 2.4, the actual v_3^h belongs to $\mathcal{B}$. If u_h is its fixed point, its block error after entry obeys

$$(\|v_n^h - u_h\|_{\mathcal{X}(I_i)})_i \leq M^{n-3}(E_3 + \rho), \quad n \geq 3. \quad (43)$$

The E_3 term is required because the entering iterate is perturbed while the rectangle is centered at unperturbed w_3 . Positivity and (41) give $M^{n-3}(E_3 + \rho) \leq (9/10)^{n-3}p$. The block intervals partition the half-line, so their squared norms add. This proves the rate in (9). Likewise,

$$\|u_h\|_{\mathcal{X}}^2 \leq \sum_i (V_i + \rho_i)^2 = C_0 < 9. \quad (44)$$

For h, g in the same ball, let $z_i = \|u_h - u_g\|_{\mathcal{X}(I_i)}$ and $\delta = A_{-1}(h - g)$. Subtracting their fixed-point equations gives $z \leq s\delta + Mz$. Since $M_{ii} < 1$, forward substitution yields $z \leq b\delta$ with b as in (42). Taking the Euclidean norm proves (10). Both endpoints must lie in the certified ball; no assumption that their difference has norm at most ε is used.

2.6 Uniform Galerkin bounds and the full equation

We finish the regularity and all-cutoff claims without presupposing global smoothness of the full equation. For $N \geq \sqrt{24}$, $P_N w_3 = w_3$. On the Galerkin subspace the map

$$T_{h,N}(f) = SP_N(W + h) - P_N Q(f, f)$$

has center defect $P_N(Sh - z_4)$. Output projection contracts all Fourier amplitude norms. The same rectangle, invariant radii and Lipschitz matrix therefore construct its global fixed point, with

$$\int_0^\infty A_0(u_{h,N})^2 dt \leq C_0 < 9. \quad (45)$$

For completeness, this finite-dimensional mild fixed point is the classical Galerkin solution. On every finite interval $[0, T]$, equivalence of the finitely many mode norms gives $P_N B(u_{h,N}, u_{h,N}) \in L^1(0, T)$ from $u_{h,N} \in \mathcal{X}(0, T)$. Variation of constants therefore gives an absolutely continuous representative satisfying

$$u'_{h,N} = -Au_{h,N} - P_N B(u_{h,N}, u_{h,N}) \quad \text{a.e.}, \quad u_{h,N}(0) = P_N(W + h).$$

This representative is continuous, so the polynomial right-hand side is continuous; hence the representative is C^1 and satisfies the ODE everywhere. Local uniqueness for the finite-dimensional polynomial ODE identifies it with the usual Galerkin solution. That solution extends globally by its energy identity, since bounded energy prevents escape to infinity in a fixed finite-dimensional space. This identification uses no global regularity premise for the full PDE.

Put $X_N = \|\nabla u_{h,N}\|_2^2$ and $Y_N = \|\Delta u_{h,N}\|_2^2$. Direct testing gives

$$\begin{aligned} \frac{1}{2}X'_N + Y_N &\leq \|u_{h,N}\|_\infty \sqrt{X_N Y_N} \leq \frac{1}{2}Y_N + \frac{1}{2}A_0(u_{h,N})^2 X_N, \\ X'_N + Y_N &\leq A_0(u_{h,N})^2 X_N. \end{aligned} \quad (46)$$

The integrating factor and (45) imply

$$\sup_{t \geq 0} X_N(t) \leq X_0 e^{C_0}, \quad \int_0^\infty Y_N(t) dt \leq X_0 e^{C_0}. \quad (47)$$

The energy equality gives $\int_0^\infty X_N dt \leq E_0/2$, hence

$$\int_0^\infty X_N(t)^2 dt \leq \frac{E_0 X_0}{2} e^{C_0} < \frac{E_0 X_0}{2} e^9 \quad \text{if } E_0 X_0 > 0. \quad (48)$$

The zero datum has zero integral.

For $N < \sqrt{24}$ the reference need not belong to the Galerkin space; (45) is not asserted. Energy instead gives

$$\sup_t X_N(t) \leq N^2 E_0 < 24 E_0, \quad \int_0^\infty X_N(t)^2 dt \leq 12 E_0^2 \leq 12 E_0 X_0.$$

The last step is the mean-zero Poincare inequality. Since $e^9 > 24$, these cutoffs also satisfy (11). This treatment does not bound the full initial energy by the critical perturbation norm.

For completeness, (47) provides uniform $L_t^\infty H^1 \cap L_t^2 H^2$ bounds on every finite time interval. The equation then bounds time derivatives in the usual spaces needed for Galerkin compactness (one may use $L_t^2 L_x^2$ here, since $H^2(\mathbb{T}^3) \hookrightarrow L^\infty$ and $\nabla u_N \in L_t^\infty L_x^2$). Simon [2], Corollary 4, applied with $H^2 \Subset H^1 \hookrightarrow L^2$ on each finite time interval, gives strong $L_t^2 H_x^1$ compactness. It passes the quadratic term to a global strong solution with the prescribed datum. The

standard local smooth solution agrees with this limit by strong uniqueness and continues for all time under the uniform H^1 bound; smooth initial data give smoothness at every finite time. The local H^1 existence, uniqueness and smoothing statements used here are reviewed in Tao [1], Proposition 2.2(i)–(ii). Its unit-torus convention transfers to our fixed torus by spatial, temporal and amplitude rescaling; we use only the qualitative result. Restarting its uniform local lifespan under the bounded H^1 norm gives continuation. Applying Fatou to the Fourier amplitudes and then time preserves the bound C_0 . The limit is therefore a mild solution in $\mathcal{X}$ and agrees with the fixed point constructed above by the $\mathcal{X}$ uniqueness argument. This proves all assertions of Theorem 2.1.

Scope of the finite certificate. The temporal partition and all reference fields are finite, whereas the perturbation and limiting equation are not truncated. The finite calculations prove the bounds used by the infinite-dimensional analytic argument. They do not establish success of repeated residual correction for arbitrary data, nor do failed larger-radius sufficient tests establish divergence or singularity. The exact arithmetic and its reproducibility record also do not constitute external mathematical peer review or a novelty claim.

3 Larger regularity neighborhoods of the amplitude segment

The convergence theorem above and the extension in this section have different conclusions. Here the second corrected reference gives larger sufficient initial-data classes for global regularity and a uniform Galerkin budget. No convergence of the ordinary heat-seeded Picard iteration is asserted on these larger classes.

In this section F, G, C, H denote the fixed spatial Fourier fields and L_1, L_5, L_6 their scalar heat profiles. The initial radius is r , the critical-error barrier is $B < 2$, and P is the reference action. We write M_{gate} for the scalar endpoint test and Γ for the total solution-action bound; the integer L is its rounded budget exponent. The scalar barrier B is distinct from the bilinear operators $B(\cdot, \cdot)$ and $B_s(\cdot, \cdot)$.

Theorem 3.1 (Regularity regions with explicit budgets). *Let $u_0 = aW + h$, where $a \in \mathbb{R}$ and h is smooth, real, mean-zero, and divergence-free. Suppose that one row of Table 1 satisfies $|a| \leq a_*$ and $A_{-1}(h) \leq r$. Then the viscosity-one unforced Navier–Stokes equation has a unique global smooth solution with datum u_0 . Writing*

$$E_0 = \|u_0\|_2^2, \quad X_0 = \|\nabla u_0\|_2^2,$$

its spherical Galerkin solutions satisfy

$$\sup_{N, t \geq 0} \|\nabla u_N(t)\|_2^2 \leq X_0 e^L, \quad (49)$$

$$\sup_N \int_0^\infty \|\Delta u_N(t)\|_2^2 dt \leq X_0 e^L, \quad (50)$$

$$B_*(u_0) := \sup_N \int_0^\infty \|\nabla u_N(t)\|_2^4 dt \leq \frac{E_0 X_0}{2} e^L. \quad (51)$$

There is no restriction on the Fourier support of h .

Write $\mathcal{C}(A, r)$ for the set of the stated smooth data admitting a decomposition $u_0 = aW + h$ with $|a| \leq A$ and $A_{-1}(h) \leq r$. Then

$$\mathcal{C}(1, 2/25) \subset \mathcal{C}(1, 1/10), \quad \mathcal{C}(9/8, 1/100) \subset \mathcal{C}(8/7, 1/100).$$

Amplitude a_*	Radius r	Error barrier B	Γ (orientation)	L
1	2/25	6/5	24.40770...	25
1	1/10	9/5	58.60140...	59
11/10	1/40	6/5	28.06654...	29
9/8	1/100	6/5	29.06160...	30
8/7	1/100	8/5	48.27248...	49

Table 1: Five certified amplitude–radius–budget triples. The first and fourth rows are nested refinements of the second and fifth, respectively, with smaller budget exponents. The displayed values of Γ , defined in (74), are for orientation; the exact rational enclosures prove $\Gamma < L$.

Thus the five rows have three inclusion-maximal data classes among the listed classes. The two nested rows retain their value through the smaller exponents 25 instead of 59 and 30 instead of 49. For a datum belonging to more than one row, one may use the smallest listed exponent for which an admissible decomposition exists. These observations do not certify a new amplitude–radius rectangle formed by mixing bounds from different rows, and no maximality among all regularity classes is asserted. The quantities E_0 and X_0 always refer to the full datum; the critical radius alone does not uniformly bound its H^1 norm.

3.1 The complete critical error inequality

Let v be a smooth finite Fourier reference and set

$$r_v = \partial_t v - \Delta v + B_s(v, v), \quad U(t) = A_0(v(t)), \quad R(t) \geq A_{-1}(r_v(t)).$$

At every cutoff containing the support of v , the exact Galerkin error $e = u_N - v$ satisfies

$$\partial_t e - \Delta e + \Pi_N\{2B_s(v, e) + B_s(e, e)\} = -\Pi_N r_v.$$

Here and below Π_N is the spherical projection. The residual is the full Fourier residual before projection; no exterior modes have been discarded when defining R .

Put $b = A_{-1}(e)$ and $d = A_1(e)$. The symmetric Leray estimate and $A_0(e)^2 \leq bd$ give, almost everywhere,

$$b' + (1 - b/2)d \leq U\sqrt{bd} + R. \quad (52)$$

Indeed, the two mixed terms contribute at most $UA_0(e)$ and the self term at most $A_0(e)^2/2$. The finite coefficient norms are absolutely continuous; the same inequalities hold at coefficient zeros by the integrated norm inequality, or by regularization and passage to the limit.

Lemma 3.2 (Endpoint gate and retained history). *Suppose*

$$b(0) \leq r < B < 2, \quad P = \int_0^\infty U(t)^2 dt < \infty, \quad R_* = \int_0^\infty R(t) dt < \infty,$$

where $R \geq 0$. Define $\Psi(s) = se^{-s/2}$ for $0 \leq s < 2$. If

$$M_{\text{gate}} := e^{P/4}\Psi(r) + \int_0^\infty \exp\left(\frac{1}{4} \int_t^\infty U(s)^2 ds\right) R(t) dt < \Psi(B), \quad (53)$$

then $b(t) < B$ for every $t \geq 0$. With

$$F_*(s) = -2s - 4\log(1 - s/2), \quad C_* = F_*(r) + \frac{B}{1 - B/2}R_*, \quad (54)$$

one moreover has

$$\left(\int_0^\infty b(t)d(t) dt \right)^{1/2} \leq \sqrt{F_*(B)} + \sqrt{F_*(B) + C_*}, \quad (55)$$

$$\int_0^\infty A_0(u_N(t))^2 dt \leq \left[\sqrt{P} + \sqrt{F_*(B)} + \sqrt{F_*(B) + C_*} \right]^2. \quad (56)$$

Proof. While $b < 2$, maximize $U\sqrt{b}\sqrt{d} - (1-b/2)d$ in $\sqrt{d} \geq 0$. As $0 < \Psi'(b) = e^{-b/2}(1-b/2) \leq 1$, this gives

$$(\Psi(b))' \leq \frac{1}{4}U^2\Psi(b) + R. \quad (57)$$

Let $J(t) = \int_0^t U^2$. Integrating (57) only up to t and multiplying by $e^{(P-J(t))/4}$ yields

$$e^{(P-J(t))/4}\Psi(b(t)) \leq e^{P/4}\Psi(r) + \int_0^t e^{(P-J(s))/4}R(s) ds \leq M_{\text{gate}}.$$

If b first reached B , the left side at that time would be at least $\Psi(B)$, contradicting (53). Finite Galerkin solutions exist globally by their kinetic-energy identity, so this first-contact argument proves $b < B$ on the entire half-line without assuming a global PDE solution. It also proves the stronger envelope

$$\Psi(b(t)) \leq \Psi(B)e^{-(P-J(t))/4}. \quad (58)$$

Define $\beta(t) \in [0, B]$ by equality in (58), using only the increasing branch of Ψ on $[0, 2)$. Then $b \leq \beta \leq B$ and

$$\beta' = \frac{U^2\beta}{4(1-\beta/2)}, \quad F'_*(s) = \frac{s}{1-s/2}.$$

The chain rule therefore proves, for every finite T ,

$$\int_0^T U^2 \frac{b^2}{(1-b/2)^2} dt \leq 4\{F_*(\beta(T)) - F_*(\beta(0))\} \leq 4F_*(B). \quad (59)$$

No strict monotonicity of J is required. In particular intervals on which $U = 0$ cause no change-of-variable problem.

Multiply (52) by $b/(1-b/2)$. After integration on $[0, T]$, drop the nonnegative terminal value $F_*(b(T))$. For $Q_T = \int_0^T bd$, monotonicity of $s/(1-s/2)$ and (59) give

$$Q_T \leq F_*(r) + \frac{B}{1-B/2}R_* + (4F_*(B)Q_T)^{1/2}.$$

The positive root of this quadratic inequality bounds $\sqrt{Q_T}$ by the right side of (55), uniformly in T . Only then take $T \rightarrow \infty$ by monotone convergence. Finally, $A_0(e)^2 \leq bd$ and Minkowski in time prove (56). $\square$

Reference-free baseline. For smooth, real, mean-zero, solenoidal data, put $r_0 = A_{-1}(u_0) < 2$, choose $r_0 < B < 2$, and take $v \equiv 0$. Then $U = R = P = 0$, and the gate is $\Psi(r_0) < \Psi(B)$, which holds because Ψ is strictly increasing on $[0, 2)$. The zero reference belongs to every Galerkin subspace, and $A_{-1}(\Pi_N u_0) \leq r_0$. Lemma 3.2, with $r = r_0$, therefore gives a finite action bound uniform in every cutoff. The enstrophy estimate and the compactness, uniqueness and continuation argument of Section 3.5 then give a global smooth solution. This is a consequence of the present comparison estimate; the threshold 2 is not attributed to the earlier small-data theorems.

3.2 The second corrected reference and its signed residual

All fields in this subsection are finite Fourier polynomials. Set

$$\begin{aligned} F &= B_s(W, W), & G &= 2B_s(W, F) = -\frac{1}{2}W + C, \\ C &= \frac{1}{2}(\sin z \cos 2y, \sin x \cos 2z, \sin y \cos 2x), & H &= B_s(F, F). \end{aligned}$$

The frequency shells of W, F, C, H have squared radii 1, 2, 5, 6, respectively. Define

$$\begin{aligned} L_1(t) &= \frac{e^{-t} - (1 + 2t)e^{-3t}}{4}, & L_5(t) &= \frac{e^{-5t} + (2t - 1)e^{-3t}}{4}, \\ L_6(t) &= \frac{(2t^2 - 2t + 1)e^{-4t} - e^{-6t}}{4}. \end{aligned} \quad (60)$$

Their initial values vanish and

$$(\partial_t + 1)L_1 = (\partial_t + 5)L_5 = te^{-3t}, \quad (\partial_t + 6)L_6 = t^2e^{-4t}.$$

Thus they are nonnegative causal heat convolutions. The reference is

$$w_{2,a} = (ae^{-t} - a^3L_1/2)W - a^2te^{-2t}F + a^3L_5C - a^4L_6H. \quad (61)$$

It starts exactly at aW and has support $|k| \leq \sqrt{6}$. For clarity, if $w_1 = ae^{-t}W - a^2te^{-2t}F$ and $z_2 = (a^3/2)L_1W - a^3L_5C + a^4L_6H$, then $w_{2,a} = w_1 - z_2$,

$$r_1 = -a^3te^{-3t}G + a^4t^2e^{-4t}H, \quad (\partial_t - \Delta)z_2 = r_1.$$

It follows algebraically that the full residual of (61) is $-2B_s(w_1, z_2) + B_s(z_2, z_2)$.

For $m = |a|$, introduce the nonnegative scalar functions

$$x = me^{-t}, \quad y = m^2te^{-2t}, \quad p = m^3L_1/2, \quad q = m^3L_5, \quad s = m^4L_6,$$

and put $K_{AB} = 2B_s(A, B)$. For $a = m \geq 0$, the complete residual is

$$\begin{aligned} r_2 &= (p^2 - 2xp)F + ypG + (x - p)qK_{WC} - (x - p)sK_{WH} \\ &\quad - yqK_{FC} + ysK_{FH} + q^2B_s(C, C) - qsK_{CH} + s^2B_s(H, H). \end{aligned} \quad (62)$$

For $a < 0$, the corresponding signed coefficients have exactly the same magnitudes as those used below. This follows by substituting the powers a, a^2, a^3, a^4 before grouping. It does not assume that the full solution is invariant under $u \mapsto -u$.

The finite Fourier identities and upper bounds required here are recorded in Table 2. Each bound is for the complete generated Fourier field with the Leray projection, not for its projection back to the original support. Their coefficient-level verification is part of the certificate appendix.

Since $p/x \leq m^2/8$, all entries in the following vector are nonnegative for $m^2 \leq 3$:

$$(f_1, \dots, f_9) = (2xp - p^2, (x - p)q, (x - p)s, yp, yq, ys, q^2, qs, s^2), \quad R_m = \sum_{j=1}^9 c_j f_j. \quad (63)$$

Consequently $A_{-1}(r_2) \leq R_m$. The subtractive terms in the first three profiles remain grouped before a Fourier norm bound is taken.

Field	A_{-1} upper cap c_j	Nonzero modes
F	17/8	12
K_{WC}	171/100	48
K_{WH}	27/20	72
G	87/40	18
K_{FC}	6/5	60
K_{FH}	183/100	120
$B_s(C, C)$	11/25	48
K_{CH}	3/4	162
$B_s(H, H)$	9/25	120

Table 2: Complete Fourier norm caps in the order used in (63).

3.3 Why the endpoint calculations cover amplitude intervals

The four shells in (61) are disjoint, and their A_0 norms are $3, 3, 3/2, \sqrt{3}$. For $m^2 \leq 8$, the fundamental coefficient retains the sign of a , since $L_1 \leq e^{-t}/4$. Hence the exact reference norm is

$$U_2(m, t) = 3me^{-t} + 3m^2te^{-2t} - \frac{3}{2}m^3(L_1 - L_5) + \sqrt{3}m^4L_6. \quad (64)$$

For $k_\lambda(t) = e^{-\lambda t} \mathbf{1}_{t \geq 0}$, direct convolution identities give

$$L_1 - L_5 = 4k_1 * k_3 * k_3 * k_5, \quad 2L_6 = 4k_4 * k_4 * k_4 * k_6.$$

Comparison of the four nonnegative factors gives $L_1 - L_5 \geq 2L_6$. Together with $0 \leq L_1 - L_5 \leq L_1 \leq e^{-t}/4$, this proves

$$U_2(m, t) \leq U_1(m, t) := 3me^{-t} + 3m^2te^{-2t} \quad (m^2 \leq 3), \quad (65)$$

$$\partial_m U_2(m, t) \geq (3 - 9m^2/8)e^{-t} \geq 0 \quad (m^2 \leq 8/3). \quad (66)$$

In the second line the omitted terms are nonnegative. Replacing $\sqrt{3}$ by a rational upper bound preserves this monotonicity, as $L_6 \geq 0$. Thus a reference-budget upper enclosure computed at a_* bounds the budget throughout $0 \leq m \leq a_*$.

The residual majorant is monotone on the same required interval. The first three profiles in (63) have the forms $m^4A - m^6D$, $m^4A - m^6D$, and $m^5A - m^7D$, respectively, with $D/A \leq 1/16, 1/8, 1/8$ wherever $A > 0$. Their derivatives have nonnegative factors bounded below by

$$4 - 6m^2/16, \quad 4 - 6m^2/8, \quad 5 - 7m^2/8,$$

all strictly positive for $m^2 \leq 3$. Where $A = 0$, the corresponding profile vanishes and the same conclusion follows by continuity. The six remaining profiles are nonnegative pure powers of m . Every amplitude endpoint in Table 1 lies in both stated ranges.

3.4 Finite exact inequalities behind the five gates

Here we specify the finite arithmetic used in (53), so that the application is not an invocation of a bare program status. A finite exponential polynomial $T(t) = \sum_{\lambda, d} c_{\lambda d} t^d e^{-\lambda t}$ with rational coefficients is integrated by

$$\mathcal{I}_n(T) := \int_0^\infty e^{-2nt} T(t) dt = \sum_{\lambda, d} c_{\lambda d} \frac{d!}{(\lambda + 2n)^{d+1}}. \quad (67)$$

Use the explicit upper enclosure

$$\sqrt{3} < s_3^+ := \frac{866025403784438647}{5000000000000000000},$$

whose square exceeds 3. Let $U_2^+(a_*, t)$ denote (64) with $\sqrt{3}$ replaced by s_3^+ . It is nonnegative and dominates $U_2(m, t)$ for $0 \leq m \leq a_*$. Set

$$P^+(a_*) = \mathcal{I}_0((U_2^+(a_*, \cdot))^2), \quad R_*(a_*) = \mathcal{I}_0(R_{a_*}), \quad I_n(a_*) = \mathcal{I}_n(R_{a_*}). \quad (68)$$

These are exact rational numbers. Although the expanded polynomial R_m has some negative coefficients, (63) proves $R_m(t) \geq 0$; therefore $0 \leq I_{n+1}(m) \leq I_n(m)$.

By (65),

$$\int_0^\infty U_1(m, t)^2 dt = \frac{9}{2}m^2 + 2m^3 + \frac{9}{32}m^4.$$

Subtracting only the heat part of the *past* U_1^2 integral gives

$$\exp\left(\frac{1}{4} \int_t^\infty U_2(m, s)^2 ds\right) \leq e^{\Delta_m} \exp(c_m e^{-2t}), \quad c_m = \frac{9}{8}m^2, \quad \Delta_m = \frac{1}{2}m^3 + \frac{9}{128}m^4. \quad (69)$$

This uses $U_1 \geq 3me^{-t}$, not a lower bound of that form for U_2 . Tonelli's theorem applies to the positive exponential series against R_m . If $t_n = c_m^n I_n(m)/n!$, then $t_{n+1} \leq c_m t_n/(n+1)$, so

$$\int_0^\infty e^{\frac{1}{4} \int_t^\infty U_2^2} R_m(t) dt \leq e^{\Delta_m} \left(\sum_{n=0}^{24} t_n + \frac{t_{25}}{1 - c_m/26} \right). \quad (70)$$

All denominators are positive at the stated endpoints.

For rational $0 \leq z < 66$ define the rational upper bound

$$\mathcal{E}(z) = \sum_{n=0}^{64} \frac{z^n}{n!} + \frac{z^{65}/65!}{1 - z/66} \geq e^z. \quad (71)$$

For a given row, replace the initial term and residual term in the gate by

$$\begin{aligned} M_{\text{gate}}^+(a_*, r) &= r\mathcal{E}(P^+(a_*)/4 - r/2) \\ &\quad + \mathcal{E}(\Delta_{a_*}) \left(\sum_{n=0}^{24} \frac{c_{a_*}^n}{n!} I_n(a_*) + \frac{c_{a_*}^{25} I_{25}(a_*)/25!}{1 - c_{a_*}/26} \right), \end{aligned} \quad (72)$$

$$\Psi^-(B) = \frac{B}{\mathcal{E}(B/2)} \leq \Psi(B). \quad (73)$$

The argument $P^+/4 - r/2$ is nonnegative in every row. Monotonicity proved above, and monotonicity of Ψ on the initial interval, imply $M_{\text{gate}} \leq M_{\text{gate}}^+$ for every datum in that row. Direct rational evaluation of (67)–(73) gives the strict inequalities recorded in Table 3. The endpoints of the data classes are closed because the gate inequalities themselves have strict margins.

For each passing row define

$$\Gamma = \left[\sqrt{P^+(a_*)} + \sqrt{F_*(B)} + \sqrt{F_*(B) + F_*(r) + \frac{B}{1 - B/2} R_*(a_*)} \right]^2. \quad (74)$$

Its strict upper comparison $\Gamma < L$ in Table 1 is another finite certificate, rather than a floating-point evaluation of logarithms. Explicitly, normalize each logarithm argument as $2^k x$ with $1 \leq x \leq 2$. For $q = (x-1)/(x+1) \in [0, 1/3]$,

$$2 \sum_{j=0}^{39} \frac{q^{2j+1}}{2j+1} \leq \log x \leq 2 \sum_{j=0}^{39} \frac{q^{2j+1}}{2j+1} + \frac{2q^{81}}{81(1-q^2)}.$$

a_*	r	B	M_{gate}^+ (orientation)	$\Psi^-(B) - M_{\text{gate}}^+$ (orientation)
1	2/25	6/5	0.636555727041...	0.02201823627...
1	1/10	9/5	0.730918119157...	0.00090726837...
11/10	1/40	6/5	0.656895071385...	0.00167889192...
9/8	1/100	6/5	0.636557679768...	0.02201628354...
8/7	1/100	8/5	0.714761998246...	0.00416434434...

Table 3: The exact numbers are the rational expressions (72)–(73), stored in the endpoint certificate. The displayed decimals are explanatory; the finite integer comparisons prove that each margin is positive.

Apply the same enclosure to $\log 2$. Square roots of the resulting nonnegative rationals are bounded above at scale 10^{30} by integer square roots with outward rounding. Substitution in (74) then compares rational numbers with the listed integers L .

The exact certificate dependencies for this section are:

Certificate	Mathematical role
<code>second_correction_fourier_check.py</code>	Fourier fields, profiles, projected products, and the nine caps in Table 2.
<code>enlarged_endpoint_gate.py</code>	Rational moments, P^+ , the full weighted residual, and the five strict endpoint inequalities.
<code>enlarged_budget_static.py</code>	Outward logarithm and square-root enclosures proving the five inequalities $\Gamma < L$.

The accompanying exact output files and hashes are supplied in the certificate appendix. The analytic implications of these finite inequalities are established here; a hash alone is not a mathematical verification of a norm estimate or a PDE passage to the limit.

3.5 All cutoffs and the full PDE conclusion

Proof of Theorem 3.1. Fix one row and its datum. At every cutoff $N \geq \sqrt{6}$, the reference (61) is retained exactly. Its initial error is $\Pi_N h$, and projection contracts both $A_{-1}(h)$ and the full residual norm. The preceding monotonicity and rational inequalities therefore verify the hypotheses of Lemma 3.2 uniformly in N . In particular,

$$\int_0^\infty A_0(u_N(t))^2 dt \leq \Gamma < L \quad (N \geq \sqrt{6}). \quad (75)$$

Write $X_N = \|\nabla u_N\|_2^2$ and $Y_N = \|\Delta u_N\|_2^2$. Testing the exact Galerkin equation against $-\Delta u_N$, using $\|u_N\|_\infty \leq A_0(u_N)$, and applying $2ab \leq a^2 + b^2$ gives

$$X'_N + Y_N \leq A_0(u_N)^2 X_N. \quad (76)$$

The integrating factor for this scalar inequality and (75) prove

$$\sup_t X_N(t) \leq X_0 e^\Gamma, \quad \int_0^\infty Y_N(t) dt \leq X_0 e^\Gamma.$$

The kinetic-energy equality gives $\int_0^\infty X_N dt \leq E_0/2$; hence $\int_0^\infty X_N^2 dt \leq (E_0 X_0/2) e^\Gamma$.

For the lower cutoffs $N < \sqrt{6}$, no comparison with a partially removed reference is needed.

Directly from energy and the spectral support,

$$\begin{aligned}\sup_t X_N(t) &\leq N^2 E_0 \leq 6X_0, \\ \int_0^\infty Y_N(t) dt &\leq N^2 \int_0^\infty X_N(t) dt \leq 3E_0 \leq 3X_0, \\ \int_0^\infty X_N(t)^2 dt &\leq \frac{N^2 E_0^2}{2} \leq 3E_0 X_0.\end{aligned}$$

Here $E_0 \leq X_0$ is the mean-zero Poincaré inequality; it does not require a small E_0 . Every listed $L \geq 2$, and $e^2 > 1 + 2 + 2 + 8/6 = 19/3 > 6$. These lower-cutoff bounds and $\Gamma < L$ prove (49)–(51) for every cutoff. We have not claimed (75) for cutoffs missing reference modes.

For completeness, the finite systems themselves exist for every time: their energy identity bounds their finite coefficient vectors and prevents finite-time escape in the polynomial ODE. On each finite time interval, (49)–(50) bound the Galerkin sequence in $L^\infty H^1 \cap L^2 H^2$. The equation bounds its time derivatives in $L^2 L^2$: by the periodic embedding $H^2 \hookrightarrow L^\infty$,

$$\|(u_N \cdot \nabla)u_N\|_2 \leq \|u_N\|_\infty \|\nabla u_N\|_2 \leq C \|u_N\|_{H^2} \|u_N\|_{H^1}.$$

Simon [2], Corollary 4, therefore gives a subsequence strongly convergent in $L^2 H^1$ on each finite interval, with the nonlinear term converging in distributions. The limit has the prescribed datum and is a global strong solution. The bounds pass by weak lower semicontinuity. Applying Fatou first to finite Fourier sums and then to their increasing limit also preserves $\int_0^\infty A_0(u)^2 dt \leq \Gamma$ for the PDE solution.

To see uniqueness in this class, take the difference of two strong solutions and test its equation in L^2 . Incompressibility eliminates one transport term, and integration by parts bounds the other by $\|u\|_\infty \|w\|_2 \|\nabla w\|_2$ for one of the solutions u . Young’s inequality and the locally integrable coefficient $\|u\|_\infty^2$ give uniqueness by Gronwall. The local smooth solution with smooth datum agrees with this strong solution. The uniform H^1 continuation bound and the local theory of [1, Proposition 2.2] extend smoothness through every finite time. No global full-PDE regularity assumption was used to obtain the Galerkin estimates. $\square$

Scope of the extension. These are sufficient regions with explicit constants, not maximal regions or an arbitrary-data assertion. They extend the initial-data classes covered by the particular reference calculation while retaining a separate proof from the ordinary-Picard convergence theorem. A failed upper gate would show only that this comparison was inconclusive. Conditional criteria, fieldwise modified-energy obstructions, and numerical experiments in other branches do not enlarge Theorem 3.1 without their own unconditional full-PDE argument.

4 A supplementary Galerkin certificate for the fixed datum

This section records a computer-assisted estimate complementary to the Picard convergence result. All norms use normalized measure on $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$, the viscosity is one, and fields are mean zero and divergence free. Write $A = -\Delta$, $B(a, b) = \mathbb{P}((a \cdot \nabla)b)$, and let P_M retain the nonzero Fourier modes with $|k| \leq M$. Denote the exact Galerkin solution from $P_M W$ by u_M , where $W = (\sin z, \sin x, \sin y)$, and put $v = u_3$.

Supplementary conclusion. The certificate and estimates below imply a unique global smooth solution u from W and

$$\|\nabla(u_M - v)(t)\|_2^2 < 10^{-3}e^{-t/2} \quad (M \geq 3), \quad (77)$$

$$\sup_M \sup_{t \geq 0} \|\nabla u_M(t)\|_2^2 < 14, \quad B_* := \sup_M \int_0^\infty \|\nabla u_M(t)\|_2^4 dt < \frac{21}{2}. \quad (78)$$

The error bound also holds for the PDE limit u . The corresponding norm bound is $\|\nabla(u - v)(t)\|_2 < 10^{-3/2}e^{-t/4}$. Separately, if $h \in H^1$ is mean zero and divergence free and $\|\nabla h\|_2 \leq 1/2500$, the datum $W + h$ generates a unique global strong solution, smooth for positive time, with

$$\|\nabla(u - v)(t)\|_2^2 < \frac{1}{500}e^{-t/2}. \quad (79)$$

Smooth perturbations retain smooth initial regularity. This ball has no Fourier-support restriction. The fixed- W energy estimates (78) are not assigned unchanged to perturbed initial data.

4.1 The error estimate and scalar comparison

The symbols $G(t)$ and $L(t)$ below are scalar functions local to this section; the scalar tail coefficient $H = \sqrt{660E}$ is defined later. They are distinct from the finite Fourier fields and profile notation of the preceding sections.

For a smooth reference field v with residual $r = v' + Av + B(v, v)$ and a smooth exact solution on its interval of existence, set $e = u - v$ and $z = \|\nabla e\|_2^2$, $D = \|\Delta e\|_2^2$, and $G = \|\nabla v\|_{L^\infty}$, using the Frobenius norm pointwise. The vector/tensor Sobolev estimate needed here is

$$\|f\|_6 \leq C_S \|\nabla f\|_2, \quad C_S = 25/3, \quad (80)$$

for componentwise mean-zero fields. To check its normalization, on a period- L torus coordinate variation gives, for nonnegative scalar a , $a \leq g_i$, where g_i is the coordinate average of $a + (L/2)|\partial_i a|$. Repeated Cauchy–Schwarz applied to $a^{3/2} \leq (g_1 g_2 g_3)^{1/2}$, followed by the arithmetic–geometric mean inequality, yields

$$\|a\|_{3/2} \leq \|a\|_1 + \frac{L}{6} \sum_i \|\partial_i a\|_1.$$

Apply this to $a = |f|^4$ and use Hölder and $\sum_i \|\partial_i f\|_2 \leq \sqrt{3}\|\nabla f\|_2$ to obtain $\|f\|_6 \leq \|f\|_2 + (2L/\sqrt{3})\|\nabla f\|_2$. At $L = 2\pi$, Poincaré and $1 + 4\pi/\sqrt{3} < 25/3$ prove (80). This argument applies directly to the Euclidean magnitude of vector or tensor fields. In particular it applies to ∇e , with $\|D^2 e\|_2 = \|\Delta e\|_2$.

Testing the error equation with $-\Delta e$, integrating the transport term once by parts, and using mean-zero Poincaré gives, initially with viscosity $\nu > 0$,

$$\frac{1}{2}z' + \nu D \leq Gz + G\sqrt{zD} + C_S^{3/2} z^{3/4} D^{3/4} + \|r\|_2 \sqrt{D}.$$

Allocate $\nu D/4$ to each of the last three terms. The cubic Young estimate has remainder $27C_S^6 z^3/(4\nu^3)$. Therefore

$$z' + \frac{\nu}{2}D \leq (2G + 2G^2/\nu)z + \frac{27C_S^6}{2\nu^3} z^3 + \frac{2}{\nu} \|r\|_2^2. \quad (81)$$

Set $\nu = 1$, $f = 2\|r\|_2^2$, and define

$$\begin{aligned}\Lambda(t) &= \int_0^t (2G + 2G^2) ds, & L(t) &= \Lambda(t) - t/2, \\ \eta &= d^2 + \int_0^\infty e^{-L} f dt, & I &= \int_0^\infty e^{2L} dt, & \kappa &= 27C_S^6 \eta^2 I,\end{aligned}\quad (82)$$

where $d^2 = z(0)$. Since $D \geq z$, $w = e^{-L}z$ satisfies $w' \leq (27C_S^6/2)e^{2L}w^3 + e^{-L}f$. The remaining-forcing function $V(t) = w(t) + \int_t^\infty e^{-L(s)}f(s)ds$ obeys $V(0) = \eta$, $V \geq w \geq 0$, and $V' \leq (27C_S^6/2)e^{2L}V^3$. Bernoulli comparison, whenever $\kappa < 1$, gives

$$z(t) \leq \frac{e^{L(t)}\eta}{\sqrt{1 - 27C_S^6\eta^2 \int_0^t e^{2L(s)} ds}} \leq \frac{e^{\Lambda(\infty)}\eta}{\sqrt{1 - \kappa}} e^{-t/2}. \quad (83)$$

4.2 Exact algebra, interval validation, and the infinite tail

The finite reference solves $v' + Av + P_3B(v, v) = 0$ from W , so $r = (1 - P_3)B(v, v)$. The supplied exact rational data give a ten-dimensional real invariant basis. Its Laplacian eigenvalues and squared norms are

$$(1, 2, 5, 5, 6, 6, 8, 9, 9, 9), \quad (3/2, 3/4, 3/16, 15, 1/8, 12, 6, 1566, 87/8, 3/2).$$

The first basis vectors are W and $F = (\sin y \cos z, \sin z \cos x, \sin x \cos y)$, using the notation of Section 2.3. Exact Fourier convolution of all 55 unordered basis pairs verifies full closure within the spherical cutoff, 133 nonzero quadratic ODE coefficients, and the entire exterior residual. There are 52 nonzero exterior pair fields, with 648 signed exterior modes, whose squared norm has 285 quartic terms. The retained support $\mathcal{K}$ satisfies

$$|\mathcal{K}| = 108, \quad \sum_{k \in \mathcal{K}} |k|^2 = 660. \quad (84)$$

The same exact audit verifies reality, divergence freedom, basis orthogonality, the initial datum, and cubic cancellation in the energy law. The basis, coefficients, and residual fields are part of the accompanying machine-readable proof data; the full coefficient lists are not printed here.

The accepted enclosure uses integer intervals with denominator 2^{224} , step $h = 1/128$, and Taylor order $p = 16$ for 1024 steps through $S = 8$. At each step strict inclusion $X + [0, h]F(\mathcal{B}) \subset \text{int } \mathcal{B}$ provides a continuous Picard enclosure. Taylor coefficients at X give the initial derivatives, and the same recurrence on $\mathcal{B}$ gives universal derivative remainders. For the dissipative diagonal component $x'_k = -\lambda_k x_k + N_k(x)$, endpoint propagation uses

$$e^{-\lambda_k h} X_k + \sum_{n=0}^{p-1} w_{\lambda_k n} N_{k,n}(X) + w_{\lambda_k p} N_{k,p}(\mathcal{B}), \quad w_{\lambda n} = \int_0^h e^{-\lambda(h-s)} s^n ds.$$

Alternating rational series enclose the exponentials and weights; integer square roots round outwards. Ordinary Taylor bounds enclose the whole time slab. Intersections of proved enclosures are used, never midpoint replacement; failed inclusion or an empty intersection rejects the run.

On these full slabs, $\|r\|_2^2$ is enclosed both by the exact quartic polynomial and by a midpoint value with an L^2 radius bound. For G , the first two basis fields satisfy $\|\nabla(aW + bF)\|_\infty = \sqrt{3(a^2 + b^2)}$; the remaining fields use their exact Fourier triangle bounds. For example, the first row of the first identity is bounded by $b^2 \cos^2 y \cos^2 z + (a \cos z - b \sin y \sin z)^2 \leq a^2 + b^2$; cyclic summation and equality at the origin prove the formula. Full-slab rectangle bounds integrate Λ , η , and I .

Let E enclose $\|v(S)\|_2^2$. The exact Galerkin energy law implies $\|v(S + \tau)\|_2^2 \leq Ee^{-2\tau}$. By (84),

$$G(S + \tau) \leq He^{-\tau}, \quad H = \sqrt{660E}, \quad \|r(S + \tau)\|_2^2 \leq 972E^2e^{-4\tau}.$$

Here the latter bound follows from $\|v\|_\infty \leq \sqrt{108}\|v\|_2$, $\|\nabla v\|_2 \leq 3\|v\|_2$, and contraction of the residual projection. Thus, writing $\Delta = 2H + H^2$,

$$\Lambda(\infty) \leq \Lambda(S) + \Delta, \quad \eta_{\text{tail}} \leq \frac{4}{7}972E^2e^{-L(S)}, \quad I_{\text{tail}} \leq e^{2L(S)+2\Delta}. \quad (85)$$

The finite Galerkin energy law ensures existence throughout this tail; no unknown global PDE solution is assumed here.

For $d = 0$, the reproduced finite certificate and these tails give

$$\eta < \frac{587}{10^9}, \quad I < 156400, \quad \Lambda(\infty) < \frac{176}{25}, \quad \kappa \leq 0.486226878267 < \frac{49}{100}. \quad (86)$$

The displayed decimal is an outward upper rounding. Exact rational arithmetic also verifies $27(25/3)^6(587/10^9)^2 156400 = 134726479/276480000 < 49/100$. A positive rational exponential series with a geometric remainder proves $e^{176/25} < 1150$. Consequently $1150(587/10^9)/\sqrt{51/100} < 1/1000$. The release replay documented in Appendix A checked all recorded interval quantities against the archived result exactly, except for wall-clock duration; the exact algebra and rational-margin outputs also agreed. This is a reproduction using the same interval implementation, not an independent implementation or a proof-assistant formalization.

4.3 All cutoffs, the PDE limit, and the perturbation ball

For each $M \geq 3$, both exact finite Galerkin flows exist globally by their energy identities. Since v belongs to the M -mode space, $e_M = u_M - v$ obeys (81) with residual $P_M r$ and $d = 0$. Orthogonal projection does not increase its L^2 norm, so the same certificate applies uniformly in M . Equations (83)–(86) prove (77). Moreover,

$$\|v(t)\|_2^2 \leq \frac{3}{2}e^{-2t}, \quad \|\nabla v(t)\|_2^2 \leq \frac{27}{2}e^{-2t}.$$

Writing $X_M = \|\nabla u_M\|_2^2$ gives

$$\sqrt{X_M(t)} < \sqrt{27/2}e^{-t} + \sqrt{1/1000}e^{-t/4}, \quad \alpha := (\sqrt{27/2} + \sqrt{1/1000})^2 < 14.$$

For $M < 3$, the direct spectral energy estimate gives $X_M(t) \leq (27/2)e^{-2t}$ instead. The exact energy identity gives $\int_0^\infty X_M dt \leq \|P_M W\|_2^2/2 \leq 3/4$ for every cutoff. Thus $\sup_{M,t} X_M \leq \alpha < 14$ and $B_* \leq 3\alpha/4 < 21/2$, including a strict uniform margin.

Retaining the positive $D/2$ term in (81) and integrating yields a uniform $L^2(0, T; H^2)$ bound for e_M , since the error estimate, the finite certificate, and (85) control all terms on the right. Together with the reference bounds this gives uniform $L^\infty(0, T; H^1) \cap L^2(0, T; H^2)$ control of u_M . The equation bounds $\partial_t u_M$ in $L^2(0, T; L^2)$; explicitly,

$$\|(u_M \cdot \nabla)u_M\|_2^2 \leq C_S^3 X_M^{3/2} \|\Delta u_M\|_2 \leq \frac{C_S^3}{2}(X_M^3 + \|\Delta u_M\|_2^2).$$

Compactness as in [2, Corollary 4] and strong uniqueness as in [1, Proposition 2.2] therefore produce a global solution in $C([0, T]; H^1) \cap L^2(0, T; H^2)$ for every finite T . Lower semicontinuity and time continuity transfer the reference-error bound; smooth initial data and parabolic

regularity give global smoothness. This passage uses standard local existence, continuation, and compactness theory, in addition to the finite certificate.

For the perturbation ball take $d^2 \leq 1/2500^2$, or its projected value for $u_M(0) = P_M(W + h)$. The same reference and residual give

$$\eta_h < \frac{747}{10^9}, \quad \kappa_h < 27(25/3)^6(747/10^9)^2 156400 = \frac{8080797}{10240000} < \frac{4}{5}.$$

Since $[1150(747/10^9)]^2/(1 - 4/5) < (1/500)^2$, (83) proves (79); the same uniform Galerkin construction applies. Its initial energy differs from that of W , so this conclusion is stated separately from (78).

Reproducibility and scope. The accompanying files in `checks/N3/` of `Code.zip`, namely `n3_exact_algebra_intermediate.json`, `n3_algebra_audit.py`, `n3_interval_core.py`, `n3_validated_run.py`, and `n3_check_margins.py` contain the complete finite data and executable checks. The runner `run_replay.py` and the release record `replay_receipt.json` in the same directory identify the reproduction and exact source hashes. Correctness still depends on these programs, their exact-arithmetic execution, and the written analysis above. The result concerns the fixed datum and the stated neighborhood; it implies neither arbitrary-data three-dimensional regularity nor a new precedence claim.

5 Scope, independent development and research assistance

The results concern the stated initial-data classes. The larger regularity neighborhoods do not inherit the ordinary Picard convergence conclusion unless they satisfy its separate hypotheses. The fixed- W constants in the finite-reference supplement are not asserted uniformly throughout a perturbation ball. No conclusion is made for arbitrary smooth three-dimensional data.

Independent development. According to the author’s account, the results reported here were developed independently of Silva’s work, which became known to the author during the publication process. The comparison with related work concerns mathematical overlap and differences; it does not imply that the present proofs were derived from Silva’s work. This records the author’s account of independent development and makes no claim of independently authenticated chronology or worldwide priority, and no allegation about another author’s development history.

Research assistance. The author identifies ChatGPT (OpenAI) as the primary AI research assistant and Claude (Anthropic) as the secondary assistant during the research. The assistance included mathematical investigation, development and review of arguments and code, and manuscript preparation. ChatGPT assisted in developing the mathematics and certificate programs, consolidating this manuscript, and orchestrating the verification runs documented in Appendix A. Claude provided additional mathematical and code review and reported separate checks. The mathematical work and exact arithmetic must be assessed from the arguments, programs and certificate data, not from agreement between AI systems. Neither system is listed as an author or an external referee. The human author remains responsible for the claims and for the accuracy of the final disclosure.

Code and evidence. The companion `Code.zip` supplies the proof-supporting material, certificate programs, required inputs and outputs, reproduction records, and instructions for

inspecting and replaying the reported finite checks. It supports reproducibility alongside the written analytic proof. This description does not assert prior public deposition, formal proof-assistant verification, or independent human peer review.

A Computational certificates and reproducibility

The arguments use two different finite certificates. The W12–W16 chain certifies exact Fourier identities, analytic time-profile bounds and rational inequalities for the whole-time Picard and endpoint arguments. It does not numerically integrate a Navier–Stokes trajectory. The supplementary N3 certificate encloses a finite-dimensional Galerkin reference by outward interval arithmetic and combines that enclosure with an analytic tail. The full-PDE conclusions in both cases also require the written analytic estimates and the cutoff-uniform continuation argument. A successful program exit alone does not establish these analytic implications.

A.1 Arithmetic and the mathematical objects certified

The decisive W12–W16 inequalities are evaluated with integers and exact rational numbers. Square roots and exponential or logarithmic expressions needed in the estimates are replaced by rational enclosures with explicit remainder bounds. Decimal conversions in the output are for orientation; they are not used to decide the strict certificate inequalities. In particular, the smallest positive rectangle-invariance slack in the W15 data-only audit is approximately

$$1.1507455879810338 \times 10^{-19}.$$

This displayed decimal is not the proof of positivity. The audit computes each slack as an exact rational number and tests it strictly against zero. Rounding the displayed slacks to a few decimal places would therefore destroy essential information. The rational coefficients and the complete inequality rows are retained in the accompanying JSON files.

The W15 finite checks verify the full 17-block memory terms, invariance of the rectangle, entry of the actual heat-seeded third iterate, and the estimate relative to the fixed point using the sum of the third-iterate entry error and the rectangle radius. The geometric-rate and solution-stability checks use positive rational majorants. The W16 checks certify the separate endpoint regions and their retained-history budgets. The replay wrapper also binds all nine field caps used in W16 to the freshly reconstructed W12 Fourier bounds, and checks the full formal twelve-monomial residual expansion. This last check supplements, rather than replaces, the Fourier reconstruction.

For N3, an interval is stored as an integer pair representing $[\ell/2^p, r/2^p]$. The implementation uses outward integer rounding, integer square-root bounds and rational series bounds for exponentials. Binary floating-point arithmetic does not determine the passing gate. The reproduced reference uses $p = 224$, Taylor order 16, step 1/128, and 1024 steps to time 8. The analytic continuation of the bounds from time 8 to infinity uses the finite ODE energy decay. The reference contains ten basis coordinates and 108 signed Fourier modes. Its complete exterior residual, rather than only the retained modes, enters the PDE error estimate. The accompanying rational margin check verifies, among other inequalities, $\kappa < 49/100$. The alternative 384-bit, order-32 computation retained in the larger research archive is not included in this selected bundle and was not rerun in the reproduction reported here.

A.2 Source-to-output map

Paths in this appendix are relative to the extracted companion `Code.zip`. The original research files are preserved under `proof_sources/W_research/`. The main analytic dependencies are

07_nonlinear_heat/heat_space_fixed_point.md, 09_geometric_growth/endpoint_critical_gate_audit.md, and the W12–W16 proof notes identified in SOURCE_MAP.json. For N3, the written bridge from the finite reference to the PDE is in 01_N3_W/n3_pde_scalar_audit.md; the finite enclosures and tail are described in 01_N3_W/n3_numerical_certificate.md.

Table 4 lists all fifteen distinct W programs rerun in dependency order. A directory label such as W15 means the corresponding directory below proof_sources/W_research/: W12 is 12_second_correction/, W13 is 13_iteration_convergence/, W14 is 14_unit_W_picard/, W15 is 15_perturbed_picard/, and W16 is 16_enlarged_endpoint/. For every listed program, its result filename is obtained by replacing .py with _result.json, with one exception: refined_W_block_gate_check.py writes refined_W_block_gate_result.json. Thus the table, together with this rule, specifies each source and output exactly. Archived comparison outputs remain beside their original source programs. The replay wrapper places copies of the programs and freshly generated dependencies together in working_copy/ under the chosen output directory. Individual programs are not intended to be run directly from the separated original-stage directories. Duplicate fresh outputs are omitted from this compact bundle; the release replay receipt is retained. The source map identifies all 61 selected mathematical dependencies and their unchanged original bytes. Historical headings in the original notes are retained as source evidence; the consolidated manuscript states the present results.

Table 4: Exact-arithmetic W certificate chain.

Stage	Source program	Certified component
W12	second_correction_fourier_check.py	Exact second-correction Fourier fields and caps.
W12	second_corrected_gate_static.py	Analytic second-reference gate.
W13	third_heat_defect_static.py	Complete third heat-defect profiles.
W14	fourth_heat_defect_static.py	Complete fourth heat-defect bounds.
W14	refined_W_block_gate_check.py	Whole-history block rectangle.
W14	refined_iteration_adversarial_static.py	Separate data-only rectangle audit.
W14	W_picard_rate_static.py	Unit-W geometric-rate majorants.
W15	perturbation_block_gate.py	Perturbed invariant rectangle.
W15	perturbed_picard_prefix_static.py	Actual heat-seeded finite-prefix entry.
W15	perturbed_stability_rate_static.py	Rates, solution budget and stability.
W15	perturbed_picard_adversarial_static.py	Separate data-only audit of entry and rates.
W16	enlarged_endpoint_gate.py	Five separate endpoint regularity regions.
W16	enlarged_budget_static.py	Retained-history solution budgets.
W16	enlarged_endpoint_adversarial_static.py	Separate data-only endpoint/budget audit.
W16	enlarged_class_inclusion_static.py	Inclusion of the earlier analytic classes.

The N3 input is n3_exact_algebra_intermediate.json, which records the exact basis, ODE coefficients and exterior residual. The original analytic notes are in proof_sources/W_research/01_N3_W/. The executable programs and this algebra input are in checks/N3/.

with unchanged file bytes; their original locations are recorded in `SOURCE_MAP.json`. Archived comparison outputs are in `checks/N3/archived/`. The runner generates fresh outputs in its own directory; duplicate fresh outputs are omitted here while the release receipt is retained. Table 5 gives the output map; `n3_interval_core.py` is the arithmetic and integration module used by the finite validator.

Table 5: N3 reproduction map.

Source program	Output	Role
<code>n3_algebra_audit.py</code>	<code>n3_algebra_audit_result.json</code>	Exact Fourier reconstruction.
<code>n3_validated_run.py</code>	<code>n3_validated_result.json</code>	Interval evolution, diagnostics and analytic tail.
<code>n3_check_margins.py</code>	<code>n3_rational_margin_check.json</code>	Separate rational gate and rounded bounds.
<code>run_replay.py</code>	<code>replay_receipt.json</code>	Commands, source hashes and archived comparisons.

A.3 Reproduction record and commands

The version 1.0 W replay ran on 16 October 2026, from 23:55:54 to 23:56:21 UTC (17 October, 05:25:54–05:26:21 Indian Standard Time). All fifteen programs returned success, and all fifteen result files were byte-for-byte identical to their archived counterparts. Inputs were copied without patches into a new working directory; result files were generated afresh in dependency order. The original selected source files were unchanged. The replayable commands, result hashes, assertion counts where supplied, and dependency checks are recorded in `checks/W/replay_receipt.json`.

The version 1.0 N3 replay ran on 16 October 2026, from 23:56:08 to 23:57:02 UTC (17 October, 05:26:08–05:27:02 Indian Standard Time). The algebra audit, finite validator and rational margin check all returned success. The mathematical fields of the finite result agreed exactly with the archived JSON after excluding the runtime field `elapsed_seconds`; the algebra and rational-margin JSON data also agreed exactly. This is a statement of parsed-data equality, not a claim that the entire N3 result file was byte-identical. Source hashes and comparisons appear in `checks/N3/replay_receipt.json`. Both reproductions used Python 3.12.14 and standard-library dependencies.

From the extracted bundle root, the W chain can be repeated into an absent or empty output directory with:

```
python3 checks/replay_W_chain.py \
  --source proof_sources/W_research --out replay_new_W
```

For N3, first make a separate copy of `checks/N3/`; from that copy run:

```
python3 run_replay.py
```

The N3 runner writes outputs within its own directory. A separate copy therefore preserves the packaged reproduction record. The W wrapper uses portable defaults and records logical relative locations; the scientific checker programs and archived certificates remain unchanged.

Do not disable Python assertions with optimization flags. A later execution naturally changes timestamps and wall-clock durations, so its entire receipt need not match the package manifest. Receipts record log names for a repeated run; raw execution logs and duplicate working trees are not distributed here. The supplied manuscript sources, selected proof notes, programs, original comparison certificates and retained version 1.0 receipts form the reproducibility record. Version 1.1 changes exposition and references; the scientific sources and receipts are unchanged, and no new scientific replay is claimed for this revision.

A.4 Evidence boundaries

These reproductions establish that the supplied programs, on the recorded runtime, reconstruct the archived passing certificates. They do not provide a separate implementation of the complete Fourier and interval algorithms. The data-only audit programs check important identities and inequalities separately, but do not constitute a formalization of every analytic lemma. The proof still depends on the correctness of the mathematical reduction, the implementations of the finite checks and enclosures, Python integer and rational arithmetic, and the ordinary execution environment. Standard PDE local existence, compactness, uniqueness and continuation arguments also remain part of the written proof.

The package manifest and SHA-256 digests serve file identification and integrity checking. They establish neither mathematical correctness nor an independently authenticated historical date. No formal proof-assistant verification or external human validation is represented by a passing receipt. In particular, the W16 regions do not acquire the W15 ordinary heat-seeded Picard-convergence conclusion merely because both certificate chains pass, and the fixed-W N3 bound does not extend unchanged to all perturbation regions. Neither numerical reproduction nor source hashing establishes novelty, priority, or regularity for arbitrary smooth three-dimensional data.

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