

Counting intersections of rational cyclic grids: an application of Smith normal form

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Abstract. Choose two points from equally spaced grids on a circle. How many choices put both their sum and their difference on specified target grids? A small example leads from fractions to integer congruences. We then use classical Smith normal form to obtain the exact count, carefully allowing the two input grids to have different sizes. Inclusion-exclusion handles a union of grids. The article is an exposition and application of established counting methods, not a claim of a new general theorem.

Keywords: linear congruences; Smith normal form; cyclic grids; counting; mathematical exposition.

A counting puzzle on a circle

Think of a clock face on which one complete turn counts as one unit. The top is labelled 0. A half turn is $1/2$, a quarter turn is $1/4$, and completing a whole turn brings us back to 0. We can add or subtract these positions and ignore any complete turns. For example, $3/4 + 1/2$ ends at the same position as $1/4$, and $1/4 - 1/2$ ends at the same position as $3/4$.

Now draw six equally spaced marks on one copy of the circle, and four equally spaced marks on another. Choose a position u from the six-mark circle and a position h from the four-mark circle. There are $6 \cdot 4 = 24$ possible ordered choices. The two coordinates have different roles: the first always comes from the six-mark circle and the second from the four-mark circle. We are counting these choices, not merely counting the positions that their sums reach.

Impose two rules at once. The sum $u + h$ must land on a three-mark circle, at 0, $1/3$ or $2/3$. The difference $u - h$ must land on a two-mark circle, at 0 or $1/2$. Complete turns are ignored in both rules. The choice $(0, 0)$ works. So does $(1/2, 1/2)$, because its sum makes a whole turn and its difference is zero. The choice $(0, 1/2)$ does not work: its sum is a half turn, which is not a mark on the three-mark circle.

How can we be sure we have counted every successful choice? For this small puzzle, we can list all 24 possibilities. There is also a short calculation using a common ruler. Divide the circle into twelve parts. The six-mark positions are then numbered 0, 2, 4, 6, 8, 10, and the four-mark positions are numbered 0, 3, 6, 9. Write the chosen positions as $2i$ and $3j$ ticks, with $0 \leq i < 6$ and $0 \leq j < 4$.

The three-mark target consists of ticks divisible by four, and the two-mark target consists of ticks divisible by six. Our rules therefore ask that $2i + 3j$ be divisible by four and $2i - 3j$ be divisible by six. Working in ticks has changed the labels, not the puzzle.

The first rule forces j to be even. If $j = 0$, it also forces i to be even, while the second rule makes i a multiple of three; only $i = 0$ works in the allowed range. If $j = 2$, the first rule forces i to be odd, and the second again makes it a multiple of three; only $i = 3$ works. Thus the two successful choices are precisely the ones found above.

Two general lessons emerge. First, different grid sizes can be put on a single ruler by using their least common multiple. Second, the size of that ruler is not the number of choices in each original grid. Here the ruler has twelve ticks, but the first coordinate still has only six choices and the second only four. If we replace both input grids by the whole twelve-mark ruler, we have changed the counting problem. We will keep careful track of this distinction.

For larger circles, a list of every pair becomes unwieldy. The point of a general formula is that its answer can be justified without listing those pairs individually. It must preserve the two rules simultaneously, the original input sizes and the convention that a full turn returns to the start. A familiar piece of integer linear algebra, Smith normal form, does exactly this. It rewrites a system of integer conditions into separate conditions that are easier to count. The proof below explains the connection, rather than asking the reader to treat the formula as a black box.

We will also allow the first input to come from either of two grids. Then some points may belong to both, so adding the two counts would count certain pairs twice. The correction is ordinary inclusion-exclusion: add the separate counts and subtract their shared part once. These are applications of established mathematics. The purpose is an accessible worked setting for the classical method, with particular attention to unequal input grids.

The question in mathematical notation

For a positive integer n , let

$$G_n = \{j/n + \mathbb{Z} : 0 \leq j < n\} \subset \mathbb{Q}/\mathbb{Z}.$$

Thus G_n is the equally spaced grid of n points on the rational circle, with addition taken modulo 1. A class x belongs to G_n precisely when $nx = 0$ in $\mathbb{Q}/\mathbb{Z}$.

Choose $u \in G_d$ and $h \in G_b$. How many *ordered pairs* (u, h) have both their sum in G_y and their difference in G_k ? Write this number as

$$C(d, b; y, k) = \#\{(u, h) \in G_d \times G_b : u + h \in G_y, u - h \in G_k\}. \quad (1)$$

The count is of pairs, rather than distinct sums, differences, or unordered selections of points. In particular, (u, h) and (h, u) are different pairs when their coordinates differ.

This question is a concrete setting for Smith normal form. The use of invariant factors to count solutions of linear congruences is classical; Butson and Stewart [1, §5, pp. 360–361] give the common-modulus solution count. We specialize that framework to (1) and spell out the normalization needed because the two input grids can have different sizes. The general counting theorem and the Smith normal form facts are credited to the existing literature.

The count and its proof

Let d, b, y, k be positive integers, and put

$$L = \text{lcm}(d, b), \quad \alpha = L/d, \quad \beta = L/b, \quad A = \begin{bmatrix} y\alpha & y\beta \\ k\alpha & -k\beta \end{bmatrix}. \quad (2)$$

For this rank-two integer matrix, define

$$s_1 = \gcd(y\alpha, y\beta, k\alpha, k\beta), \quad s_2 = \frac{2y\alpha\beta}{s_1}, \quad J_L(s) = \frac{L}{\gcd(L, s)}. \quad (3)$$

All gcds are positive; the sign of an entry does not affect its gcd. The integers s_1, s_2 are the Smith invariant factors of A .

Theorem 1

With the positive integer orders and definitions above,

$$C(d, b; y, k) = \frac{db}{J_L(s_1)J_L(s_2)} = \frac{db}{L^2} \gcd(L, s_1) \gcd(L, s_2). \quad (4)$$

The quotients in (4) are exact integers.

We recall the normal form used in the proof. An integer matrix is *unimodular* when its determinant is ± 1 ; its inverse is again an integer matrix. For a full-rank 2×2 integer matrix A , there are unimodular U, V such that

$$UAV = \text{diag}(s_1, s_2), \quad s_1 | s_2.$$

Here s_1 is the gcd of the entries and $s_1 s_2 = |\det A|$. These are the standard Smith normal form and determinantal-divisor facts; see Stanley [2, Theorems 2.3–2.4] for the related quotient description and the minors formula. Reduction modulo L preserves the invertibility of U, V .

Proof

Write $u = i/d + \mathbb{Z}$ and $h = j/b + \mathbb{Z}$, with $i \in \mathbb{Z}/d\mathbb{Z}$ and $j \in \mathbb{Z}/b\mathbb{Z}$. In denominator L , these are $\alpha i/L$ and $\beta j/L$. The two membership conditions in (1) become

$$y(\alpha i + \beta j) \equiv 0 \pmod{L}, \quad k(\alpha i - \beta j) \equiv 0 \pmod{L}. \quad (5)$$

Thus the admissible pairs form the kernel of

$$\phi: \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z} \longrightarrow (\mathbb{Z}/L\mathbb{Z})^2, \quad (i, j) \mapsto A \begin{pmatrix} i \\ j \end{pmatrix} \pmod{L}.$$

Multiplying the first column of A by d gives $(yL, kL)^T$, and multiplying the second by b gives $(yL, -kL)^T$. Therefore ϕ is well defined. Moreover, any integer coefficient of the first column may be reduced modulo d , and any coefficient of the second modulo b , without changing the image. Consequently

$$\text{im } \phi = \text{im } \bar{A}, \quad \bar{A}: (\mathbb{Z}/L\mathbb{Z})^2 \longrightarrow (\mathbb{Z}/L\mathbb{Z})^2. \quad (6)$$

This equality is the point at which the rectangular input domain is related to the square matrix domain.

Since $\det A = -2y\alpha\beta \neq 0$, the matrix has rank two. The gcd and determinant give precisely (3). In particular, every entry is divisible by s_1 , so s_1^2 divides $\det A$ and $s_1|s_2$. The unimodular transformations giving Smith normal form preserve the image size modulo L .

Multiplication by s on $\mathbb{Z}/L\mathbb{Z}$ has kernel of size $\gcd(L, s)$ and hence image of size $J_L(s)$. To see the kernel size directly, write $L = gL'$ and $s = gs'$ for $g = \gcd(L, s)$; then $sx \equiv 0 \pmod{L}$ is equivalent to $L'|x$, giving g residues modulo L . Applying this to the two Smith coordinates and using (6),

$$|\operatorname{im} \phi| = J_L(s_1)J_L(s_2).$$

The finite-group first isomorphism theorem now gives

$$|\ker \phi| = \frac{|\mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z}|}{|\operatorname{im} \phi|} = \frac{db}{J_L(s_1)J_L(s_2)}.$$

This proves the count and its exact integrality. No floor operation is involved.

The exact classical relationship

The same formula can be read directly from classical square-domain congruence counting. Define the surjective reduction map

$$\tau : (\mathbb{Z}/L\mathbb{Z})^2 \rightarrow \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z}, \quad (x, z) \mapsto (x \bmod d, z \bmod b).$$

Every fiber of τ has $(L/d)(L/b) = L^2/(db)$ elements, and the column-period calculation gives $\bar{A} = \phi \circ \tau$. Therefore

$$|\ker \bar{A}| = \frac{L^2}{db} |\ker \phi|. \tag{7}$$

For a consistent common-modulus integer system of rank r in n unknowns, the classical count in [1, §5] is

$$L^{n-r} \prod_{i=1}^r \gcd(L, s_i).$$

Our system is homogeneous and thus consistent, with $n = r = 2$. Its square-domain count is $\gcd(L, s_1)\gcd(L, s_2)$. Combining that count with (7) gives exactly (4).

Butson and Stewart use row-vector notation; taking a transpose to match our column-vector convention does not change the invariant factors. The rational-grid notation and the fiber normalization here are an explicit specialization, rather than notation attributed to their paper. This also makes clear why the present count is an application of classical theory.

Replacing a grid by a union

For positive integers a, b, y, k, r , define

$$Q(a, b; y, k; r) = \#\{(u, h) \in (G_a \cup G_r) \times G_b : u + h \in G_y, u - h \in G_k\}.$$

Each pair is counted once, including those whose first coordinate belongs to both grids.

Corollary 1

The union count is

$$Q(a, b; y, k; r) = C(a, b; y, k) + C(r, b; y, k) - C(\gcd(a, r), b; y, k). \quad (8)$$

Each occurrence of C uses its own $L, \alpha, \beta, s_1, s_2$.

Proof

Let $g = \gcd(a, r)$. A class annihilated by both a and r is annihilated by g , by Bezout's identity. Conversely, a class annihilated by g is annihilated by a and r . Hence $G_a \cap G_r = G_g$. The intersection of the two admissible ordered-pair sets is exactly the admissible set over $G_g \times G_b$. Ordinary inclusion-exclusion proves (8).

If $a = r$, the formula reduces to a single-grid count. If $a|r$, then $G_a \subseteq G_r$, and it reduces to $C(r, b; y, k)$; the analogous statement holds when $r|a$. Unit orders are included because $G_1 = \{0\}$.

Three worked examples

Unequal denominators

Take $(d, b; y, k) = (6, 4; 3, 2)$. Then $L = 12$, $\alpha = 2$, $\beta = 3$, and

$$A = \begin{bmatrix} 6 & 9 \\ 4 & -6 \end{bmatrix}, \quad (s_1, s_2) = (1, 72), \quad (J_{12}(s_1), J_{12}(s_2)) = (12, 1).$$

Thus $C(6, 4; 3, 2) = 24/12 = 2$. The two pairs are $(0, 0)$ and $(1/2, 1/2)$, with coordinates interpreted modulo 1. The square-domain kernel has size $\gcd(12, 1)\gcd(12, 72) = 12$, but the input domain has only $6 \cdot 4 = 24$ points. Dividing by the fiber size $12^2/24 = 6$ gives the required 2 pairs. Using the square-domain kernel count without this normalization would give the wrong answer.

Parity on an even grid

Take $(d, b; y, k) = (6, 6; 3, 3)$. Here

$$L = 6, \quad A = \begin{bmatrix} 3 & 3 \\ 3 & -3 \end{bmatrix}, \quad (s_1, s_2) = (3, 6), \quad (J_6(s_1), J_6(s_2)) = (2, 1).$$

The formula gives $C(6, 6; 3, 3) = 36/2 = 18$. Directly, for $u = i/6$ and $h = j/6$, the conditions say that $i + j$ and $i - j$ are even. This is equivalent to i, j having the same parity. There are $3 \cdot 3$ even-even and $3 \cdot 3$ odd-odd choices. This example shows why dividing freely by 2 in congruences with an even modulus would be invalid.

A union with a shared grid

Take $a = 4$, $r = 6$, $b = 4$, $y = 3$, $k = 6$. The three single-grid computations are shown in Table 1.

d	L	α	β	(s_1, s_2)	$(j_L(s_1), j_L(s_2))$	$C(d, 4; 3, 6)$
4	4	1	1	(3, 12)	(4, 1)	4
6	12	2	3	(3, 72)	(4, 1)	6
2	4	2	1	(3, 24)	(4, 1)	2

Table 1. The three single-grid counts for the union example.

Since $\gcd(4, 6) = 2$, the union contains $4 + 6 - 2 = 8$ admissible pairs. A direct list, modulo 1, is

$$(0, 0), (1/4, 3/4), (1/2, 1/2), (3/4, 1/4), \\ (1/3, 0), (2/3, 0), (1/6, 1/2), (5/6, 1/2).$$

The two shared pairs $(0, 0)$ and $(1/2, 1/2)$ must be subtracted once.

Contribution and scope

The contribution of this exposition is the explicit cyclic-grid formulation, the careful connection between rectangular and square domains, and examples showing how to use the resulting exact formula. Smith normal form, the classical congruence count, and inclusion-exclusion are established mathematics. We claim no new general congruence-counting theory and make no priority claim for this particular notation or specialization.

The use of a common period does not contradict the Chinese remainder theorem. Decomposing the full congruence system at coprime factors of L remains valid; the Smith computation is a convenient global calculation. No failure of primewise counting is asserted.

All grid orders appearing above are positive integers. The note concerns unshifted finite grids in $\mathbb{Q}/\mathbb{Z}$ and counts ordered pairs satisfying the stated membership conditions. Zero, negative, or nonintegral orders are outside the definitions. The examples illustrate the proof; they do not replace it. The claims are limited to the exact counts and mathematical arguments presented here.

Author contribution

The author reports that he proposed the motivating hypothesis and provided logical and scientific direction for this exposition, with AI assistance. This contribution statement makes no originality or priority claim for the classical mathematics used here.

AI assistance and review status

ChatGPT (OpenAI) provided the main AI assistance with generative drafting, editing and reorganization, mathematical inspection, reference checks, and examples. Codex assisted the present manuscript preparation. The author reports limited early-stage assistance from Claude (Anthropic); its specific contribution to this note and the exact historical model versions are not established by the retained records. This assistance was not limited to language correction. The author reports having read the manuscript. Mathematical inspection was AI-assisted; completed personal verification of the mathematical proof is not claimed. No independent human peer review or proof-assistant kernel verification is claimed.

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